ON THE ELECTROMAGNETIC INTERACTIONS OF ELEMENTARY PARTICLES

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

By

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in the School of Physics

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March, 1974

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ACKNOWLEDGMENTS

I wish to extend thanks to my advisor, Dr. Helmut J. Biritz, for guidance in the selection of and help during the course of this work. I also extend gratitude to Dr. Rudolf M. Ahrens and Dr. Marvin B. Sledd for their many helpful comments and suggestions concerning the preparation of the final manuscript.

I thank my parents, Mr. and Mrs. Roy E. Landers, for their continued support and faith; and special thanks are due my wife, Carol, and daughter, Kimberly, not only for support and encouragement, but for their bearing of the burden of life with a student.

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SUMMARY

The quantum field operators used in the description of the interactions of elementary particles are not uniquely determined by Lorentz invariance requirements. Rather, these requirements restrict the fields which may be used to certain classes of fields dependent on the mass and spin of the particle to be described. To determine which field may be used for a particular particle it is necessary to compare theoretical results with experimental data involving that particle. The interaction of massive, spin 1/2 particles with the electromagnetic vector potential is investigated in this work. Fields which transform under Lorentz transformations according to finite-dimensional, irreducible representations of SL(2,c) are chosen from the class of fields available for massive, spin 1/2 particles. A current operator is constructed from these fields and its matrix elements yield specific functions for the form factors of the particle. Form factors calculated in such a manner are in the form of polynomials in the momentum transfer; and unless several such polynomials are combined, the only particles which are described correctly are electrons and muons.

Chapter I presents an introduction to the problem. The general expression for the matrix element of the vector current and the Rosenbluth formula are presented along with a short discussion of various previous attempts at the explanation of the nucleon form factors. The symmetry properties required of an electromagnetic current are

presented in Chapter II along with the fields which transform under Lorentz transformations according to the finite-dimensional, irreducible representations of SL(2,c). The manner in which such fields are coupled to form a vector operator is also presented in Chapter II, which ends with the formation of a general expression for the electromagnetic current operator. In Chapter III the matrix elements of the current operator are found and the form factors extracted. A discussion of the results of the calculation is in Chapter IV.

CHAPTER I

INTRODUCTION

The interactions and properties of elementary particles are studied by observation of scattering processes and decays. Information obtained from such observations is in the form of scattering cross sections and transition probabilities. In this work, elastic scattering of elementary particles via the electromagnetic interaction is considered. Only massive particles with spin 1/2 are considered specifically, but the formalism is developed in a general enough manner so that applications can be made to massive bosons or higher spin fermions in an evident way. The restriction to massive particles reflects the fact that there is known only one massless particle which takes part in the electromagnetic interaction, and this particle, the photon, is itself the quantum of the electromagnetic field. The restriction to spin 1/2 particles is motivated partly by calculational convenience, but primarily by the fact that the electron, the muon, the proton, and the neutron, which are among the most studied particles, are all spin 1/2 particles.

Coulomb Interaction

Before considering the relativistic calculations involved in the consideration of electromagnetic interactions, it is instructive to examine the nonrelativistic domain. This provides an introduction, at an elementary level, to some of the methods and constructs which appear in the more general relativistic case.

The Coulomb interaction may be expressed in terms of interaction energy in the following way: the electrostatic interaction energy between two bodies, separated by a distance large in comparison with their dimensions, varies directly with the product of their charges and inversely with their separation. The form of the interaction energy is*

$$\frac{q_1\, q_2}{4\pi r} \tag{I-1}$$

for point charges q_1 and q_2 separated by any distance r, or in general for charge distributions which are separated from each other sufficiently. For the general case of charge distributions described by the charge densities $\rho_1(\vec{r}_1)$ and $\rho_2(\vec{r}_2)$, the experimentally observed validity of the superposition principle allows the mutual interaction energy to be written in the form

$$\frac{1}{4\pi} \int d\vec{r}_1 d\vec{r}_2\; \rho_1(r_1)\, \frac{1}{|\vec{r}_1 - \vec{r}_2|}\, \rho_2(\vec{r}_2) \tag{I-2}$$

from which (I-1) is recovered for point charges, whose charge densities are Dirac δ functions. When one of the charges is a point charge (e.g. $\rho_1(\vec{r}_1) = q_1\delta(\vec{r}-\vec{r}_1)$), one of the integrations in (I-2) may be performed; and the interaction energy becomes

*The units, notation, and conventions used in this work are discussed in Appendix A.

$$q_1 \, A_0(\vec{r})$$

where

$$A_o(\vec{r}) = \frac{1}{4\pi} \int d\vec{r}_2 \, \frac{\rho_2(\vec{r}_2)}{|\vec{r}-\vec{r}_2|}$$

is the potential with which the charge q_1 interacts.

As an introductory example and an aid to understanding, elastic scattering of a point charge from a fixed potential is considered. The more complicated case of two nonpoint charge distributions scattering from each other is not considered here. However, when the relativistic scattering processes are considered later, the effect of using an extended charge as a probe will be evident in first-order perturbation approximations. This will be the case since in the first approximation the two charge currents, that of the probe and that of the target, are connected only by the photon propagator, and their matrix elements can be separately considered.

We now apply first-order perturbation theory (the Born approximation) to the elastic scattering of a point charge -e from a fixed potential

$$A_o(\vec{x}) = \frac{Ze}{4\pi} \int d\vec{x}' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \qquad \text{(I-3)}$$

in which $Ze\rho(\vec{x})$ is the charge density of the target, and $\rho(\vec{x})$ is normalized according to

$$\int d\vec{x} \, \rho(\vec{x}) = 1 \qquad \text{(I-4)}$$

The projectile has initial momentum $\vec{p}$ and final momentum $\vec{p}'$, with $|\vec{p}| = |\vec{p}'| = p$ and $\vec{p}\cdot\vec{p}' = p^2\cos\theta$. The interaction energy is

$$U(\vec{x}) = -eA_o(\vec{x}) = \frac{-Ze^2}{4\pi} \int d\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

or

$$U(\vec{x}) = -Z\alpha \int d\vec{x}' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \qquad \text{(I-5)}$$

where $\alpha = \frac{e^2}{4\pi} \simeq 1/137$ is the fine structure constant. The Born approximation, good for low Z and projectile energies high relative to bound state energies, gives[1]

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi}\right)^2 \left| \int d\vec{x}\, e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}}\, U(\vec{x}) \right|^2 \qquad \text{(I-6)}$$

for the differential scattering cross section. The integral in (I-6), by use of (I-5), is

$$-Z\alpha \int d\vec{x}\, d\vec{x}'\, e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}} \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= -Z\alpha \int d\vec{x}'\, e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}'} \rho(\vec{x}') \int d\vec{x}\, \frac{e^{-i(\vec{p}'-\vec{p})\cdot(\vec{x}-\vec{x}')}}{|\vec{x}-\vec{x}'|} \qquad \text{(I-7)}$$

After multiplying and dividing (I-7) by $|\vec{p}'-\vec{p}|^2$, the relation

$$|\vec{p}'-\vec{p}|^2\, e^{-i(\vec{p}'-\vec{p})\cdot(\vec{x}-\vec{x}')} = -\nabla^2\, e^{-i(\vec{p}'-\vec{p})\cdot(\vec{x}-\vec{x}')}$$

can be used, after which the second integral in (I-7) can be integrated by parts and the identity

$$-\nabla^2 \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = 4\pi\delta(\vec{x}-\vec{x}')$$

used to yield finally

$$\int d\vec{x}\, e^{-i(\vec{p}'-\vec{p})\cdot\vec{x}}\, U(\vec{x}) = \frac{-4\pi Z\alpha}{|\vec{p}'-\vec{p}|^2} F(\vec{p}'-\vec{p}) \tag{I-8}$$

where

$$F(\vec{k}) = \int d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\, \rho(\vec{x}) \tag{I-9}$$

is a form factor characterizing the target charge distribution. The normalization condition (I-4) implies

$$F(0) = 1 \tag{I-10}$$

Use of (I-8) and (I-9) in (I-6) shows that

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha m}{2\ p^2 \sin^2\frac{\theta}{2}}\right)^2 |F(\vec{p}'-\vec{p})|^2 \tag{I-11}$$

For a point source $\rho(\vec{x}) = \delta(\vec{x})$ and $F(\vec{p}'-\vec{p}) = 1$ for all values of $\vec{p}'-\vec{p}$. In this case (I-11) reduces to the Rutherford formula.

The form factor $F(\vec{k})$, in which $\vec{k} = \vec{p}'-\vec{p}$ is the momentum transfer, is the Fourier transform of the charge distribution. With the representation of the δ function

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{-i\vec{k}\cdot\vec{x}}$$

(I-9) may be solved for the normalized charge density

$$\rho(\vec{x}) = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot\vec{x}}\, F(\vec{k}) \tag{I-12}$$

so that knowledge either of the charge density or of the form factor is equivalent to knowledge of the other. For a spherically symmetric charge density the form factor is real; and measurements of cross sections provide the means of obtaining the form factor, by use of (I-11) in the form

$$|F(\vec{k})|^2 = \frac{d\sigma}{d\Omega} \Big/ \left.\frac{d\sigma}{d\Omega}\right)_{Ruth}$$

The sign of $F(\vec{k})$ must be determined from other information. In the case of a proton the charge is known to be positive, which leads to the requirement $F(0) = 1$, whereas for an antiproton the requirement is $F(0) = -1$. The value of $F(0)$ is a normalization condition on the charge density. If the exponential in (I-9) is expanded, the result is

$$F(\vec{k}) = \int d\vec{x}\, \rho(\vec{x}) \left[1 - i\vec{k}\cdot\vec{x} - \frac{(\vec{k}\cdot\vec{x})^2}{2!} + \ldots \right] \tag{I-13}$$

a series expansion of $F(\vec{k})$ in terms of $\vec{k}$. For a spherically symmetric density, the terms involving odd powers of $\vec{k}\cdot\vec{x}$ do not contribute to $F(\vec{k})$, and (I-13) becomes

$$F(k) = \int d\vec{x}\, \rho(r) - \frac{1}{2} k^i k^j \int d\vec{x}\, \rho(r)\, x^i x^j + \ldots \tag{I-14}$$

The second integral in (I-14), by symmetry, is

$$\int d\vec{x}\,\rho(r)\,x^i x^j = \tfrac{1}{3}\delta ij \int d\vec{x}\,\rho(r)\,r^2$$

so (I-14) becomes

$$F(k) = \int d\vec{x}\;\rho(r) - \frac{k^2}{6}\int d\vec{x}\;\rho(r)\,r^2 + \ldots \qquad \text{(I-15)}$$

from which evidently F(0) is the normalized charge and $6\left.\frac{dF}{d(-k^2)}\right|_{k=0}$ is the mean square radius of the charge distribution. In fact, (I-15) is an expression for F(k) in powers of k^2, the expansion coefficients being essentially the moments of the charge distribution

$$\int d\vec{x}\;\rho(r)\;r^{2n}$$

The above example has served to introduce the concept of the charge form factor in a straightforward manner. The connection between the form factor and the charge density was seen to be that one is the Fourier transform of the other. While the charge density is perhaps a more intuitive physical description of a system than is the form factor, the charge densities of elementary particles are not directly observable by conventional means. Rather, scattering cross sections and decay lifetimes provide the information about particles, and any conclusions concerning charge densities must be deduced from these data. The form factor of a particle seems to be the more natural physical description of its charge, in the sense that it may be measured by comparing observed cross sections with those calculated for point charges. The charge density description of the particle may then be obtained by the Fourier transform of the form factor. All these considerations are based on the perturbation techniques used in the calculations.

Before giving considerations to the relativistic, quantum mechanical treatment of electromagnetic scattering processes, it is worthwhile to discuss the importance of the probe, the projectile in the example discussed. When the probe is a structureless point charge, the considerations discussed previously apply. However, several questions arise in this connection. First, what is the form of the cross section for a probe which has a nonpoint charge distribution? Second, how does one know whether or not the probe is a point charge? The answers to these questions occur quite naturally together, and they are considered in more detail later. Here we merely discuss the results briefly. As mentioned already, the electromagnetic scattering of two particles, in the first approximation, is described by the interaction of the currents of the particles via photon exchange. The matrix element of the interaction Hamiltonian involves the separate matrix elements of the two currents, connected by the photon propagator. Each of these matrix elements has form factors characteristic of the particle. In this way, each particle contributes its own form factor, and the form factors simply multiply together. This effectively answers the first question, and leads to the answer to the second as well. To determine whether a particle is a point particle, at least to within the approximations used and the energy limitations of the particle accelerators, one calculates the scattering cross section for the scattering of two of the particles in question, assuming each to be a point particle, and compares this with experimentally determined cross sections. Good agreement is an indication that the particle acts as a point particle.

In the calculation of the scattering of identical particles, the indistinguishability of the particles as well as the statistics obeyed by the particles must be taken into account. For electron-electron and proton-proton scattering, the particles are identical; and since in both cases the particles have spin 1/2, the Pauli exclusion principle must be obeyed. The scattering cross section for pure Coulomb scattering of point charge, spin 1/2, identical particles is referred to as the Mott cross section.

Classical calculations equating mass to the inertia of the electromagnetic field surrounding a charged particle indicate that a particle with charge $\pm$ e and mass m has a "radius" on the order of α/m. The classical charge radius of the electron is about 2.8f (1f = 1 fermi = 10^{-13}cm), while that of the proton is approximately 1.5×10^{-3}f. These classical considerations indicate that deviations from the Mott cross section for electrons are not expected until the electrons have a classical closest approach of about 5f, while protons should not deviate from the Mott cross section until they approach each other to within about 3×10^{-3}f. In order for two electrons to approach each other to within 5f, they must have a total center of mass kinetic energy of about 0.3MeV when they are far apart, which makes them relativistic. In that case, rather than trying to use relativistic potentials, we use quantum electrodynamics (QED). QED is briefly discussed in the next section; here we simply say that it is a formalism for the relativistic, quantum mechanical counterpart of the classical Coulombian electrodynamics.

The electron (as well as the muon) presently gives all indications of being a point particle, in that the calculations of QED which assume a point electron agree with experiment[2] in cases corresponding to classical distances of closest approach well below the classical electron radius. However, the same is not true for the proton. Early results[3] of proton-proton scattering showed deviations from the Mott cross section for projectile kinetic energies equal to or greater than approximately 0.6MeV. The 0.6MeV energy indicates nonrelativistic protons which have a classical closest approach of about 5f, several orders of magnitude larger than the classical proton radius. These deviations from Mott scattering might be due to a breakdown of Coulomb's law at small distances, to nonpoint charge protons, to short-range interactions, or to any combination of these. Certainly protons do take part in the short-range, strong interactions, and this is enough to qualitatively explain the deviations. For this reason, protons can not be effectively used to probe the electromagnetic structure of nuclei until the strong interactions are understood better. Until then, the best probe for studying the electromagnetic structure of elementary particles (as well as nuclei) is the electron, which does not participate in the strong interactions.

Relativistic Electron Theory

The indications mentioned above that the electron is a point particle are discussed later in this section; for the present, a point electron is assumed. How, then, can such a probe be used to study nuclear and particle charge distributions?

The details of the use of the electron as a probe in the study of particle charge distributions can only be discussed after the relativistic theory of the electron has been considered. The general method in which electrons are used as probes consists of scattering processes. In scattering experiments using beams of electrons with sharply defined energies, we can anticipate the effects due to the wave aspects of the electron. For example, in the scattering of electrons from a charge distribution, if the electron wavelength is large compared to the charge distribution, diffraction effects will make the scattering appear to be from a point charge. If the finite size of a distribution is to be "observed" by an electron, then the electron wavelength must be on the order of magnitude of the size of the distribution. If the details of the distribution are in question, electrons with wavelengths small in comparison with the distribution must be used.

Typical nuclear dimensions range between approximately 1f and 10f. An electron with a wavelength of 10f has an energy of roughly 120MeV, while a 1f wavelength electron has an energy of about 1.2GeV. These figures indicate that electrons used to explore nuclear structure must be extremely relativistic. The relativistic formulation of the electromagnetic interactions of electrons is therefore a necessity in the study of nuclear and particle structure by means of electron scattering.

Electrodynamics may be cast into a covariant form in which

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

defines the field strengths from the potential and Maxwell's equations have the form

$$\partial_\alpha F^{\alpha\beta} = J^\beta \tag{I-16}$$

$$e_{\alpha\beta\mu\lambda}\, \partial^\beta\, F^{\mu\lambda} = 0 \tag{I-17}$$

where $J^\mu = (\rho,\vec{J})$, and ρ and $\vec{J}$ are the source charge and current densities. The covariant form of the Lorentz force law is

$$f_\mu = F_{\mu\lambda}\, J^\lambda$$

In the Lorentz gauge $\partial_\mu\, A^\mu = 0$; and (I-16) can be written

$$\Box A^\mu = J^\mu \tag{I-18}$$

where the definition

$$\Box = \partial^\mu \partial_\mu$$

is used. The success of classical electrodynamics suggests that the quantum equations, when possible, be modeled after the classical ones. Such a correspondence principle is not available for spin 1/2 particles, and the first successful theory for these particles, Dirac's relativistic theory of electrons, is still the most widely used.

The Dirac equation for a free, spin 1/2 particle of mass m may be written

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \qquad \text{(I-19)}$$

and the adjoint equation is

$$i\partial_{\mu}\overline{\psi}(x)\gamma^{\mu} + m\overline{\psi}(x) = 0 \qquad \text{(I-20)}$$

where $\overline{\psi}(x) = \psi^{\dagger}(x)\gamma^{0}$ is the adjoint field. That the current

$$j^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x) \qquad \text{(I-21)}$$

is conserved--i.e., that

$$\partial_{\mu}\, j^{\mu}(x) = 0 \qquad \text{(I-22)}$$

follows from (I-19) and (I-20). Use of (I-21) as the current in (I-18) certainly will not work, since (I-21) resulted from a free particle theory, in which the particles are not yet coupled to the electromagnetic field. The conventional coupling of the particles to the electromagnetic field is referred to as "minimal coupling." There is no unique prescription for the coupling, but a look at the classical electrodynamics of particles will prove helpful in providing a plausible explanation for the minimal coupling scheme in quantum mechanics.

The Lagrangian for a particle with charge q and mass m in an electromagnetic field can be written

$$L = \frac{m\,\vec{v}\cdot\vec{v}}{2} - q\phi + q\vec{A}\cdot\vec{v}$$

from which the canonical momentum $\vec{p}$ conjugate to the position $\vec{x}$ is obtained:

$$p^i = \frac{\partial L}{\partial v^i} = mv^i + qA^i$$

The Hamiltonian for the system is

$$H = p^i v^i - L = \frac{(\vec{p}-\vec{q}A)\cdot(\vec{p}-\vec{q}A)}{2m} + q\phi \qquad \text{(I-23)}$$

The Hamiltonian for a particle moving in a velocity-independent potential with a potential energy V is

$$H = \frac{\vec{p}\cdot\vec{p}}{2m} + V \qquad \text{(I-24)}$$

and (I-23) can be obtained from (I-24) by replacing V with the electrostatic potential energy $q\phi$ and the canonical momentum $\vec{p}$ with $\vec{p}-\vec{q}A$.

For the relativistic case, we first consider a free particle. The four-velocity of the particle is defined as the proper-time derivative of the position

$$U^\alpha = \frac{dx^\alpha}{d\tau}$$

and a suitable invariant Lagrangian is

$$L = \frac{m}{2}\,U^\alpha U_\alpha$$

The canonical momentum is

$$p^{\alpha} = \frac{\partial L}{\partial U_{\alpha}} = mU^{\alpha}$$

and the invariant Hamiltonian is

$$H = \frac{p^{\alpha} p_{\alpha}}{2m} \tag{I-25}$$

If the particle moves in an electromagnetic field, a suitable invariant Lagrangian is

$$L = \frac{m}{2} U^{\alpha} U_{\alpha} + q\ U^{\alpha} A_{\alpha}$$

The canonical momentum resulting from this Lagrangian is

$$p^{\mu} = \frac{\partial L}{\partial U_{\mu}} = m\ U^{\mu} + q\ A^{\mu} \tag{I-26}$$

and the Hamiltonian is

$$H = p^{\alpha} U_{\alpha} - L = \frac{m}{2} U^{\alpha} U_{\alpha}$$

which upon inserting (I-26) becomes

$$H = \frac{(p^{\alpha} - qA^{\alpha})(p_{\alpha} - qA_{\alpha})}{2m} \tag{I-27}$$

Equation (I-27) may be obtained from (I-25) by replacing p^{μ} with $p^{\mu} - qA^{\mu}$.

From the above discussion we anticipate that a prescription for the transition from free relativistic mechanics to relativistic electrodynamics might be the replacement of p^μ with $p^\mu - qA^\mu$ in the presence of an electromagnetic field. In quantum mechanics p^μ is the operator

$$p^\mu = i\partial^\mu$$

and the "minimal coupling" scheme consists of the replacement*

$$\partial^\mu f(x) \rightarrow (\partial^\mu + iqA^\mu)\ f(x)$$

So the quantum mechanical equations of motion for the electromagnetic interactions of particles are obtained from the equations of motion for free particles by the minimal coupling scheme, which consist of the replacement

$$\partial_\mu \rightarrow \partial_\mu + iqA_\mu \tag{I-28}$$

This discussion has not been intended as anything other than a plausibility argument for the minimal coupling scheme for QED,** which

---

*For operation on the complex conjugate function, as in the case of the electromagnetic interactions of pi mesons for example, the prescription is

$$\partial_\mu\ f^* \rightarrow (\partial_\mu - iqA_\mu)f^*$$

**The minimal coupling scheme is also suggested by the requirement that the Lagrangian be invariant under space-time dependent phase transformations of the fields, in combination with gauge invariance of the second kind of electromagnetic potentials (see, for example, Gasiorowicz[7]).

does in fact seem to be the correct coupling for electrons and muons to the electromagnetic field. Though other couplings to the electromagnetic field may be possible, none is known which is as successful and simple as minimal coupling.

The Dirac equation and its adjoint follow from the free particle Lagrangian density

$$\mathcal{L}_0(x) = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{I-29}$$

by use of the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0$$

If the minimal coupling is used in the Lagrangian (I-29), there results

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{em} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) - qA_\mu(x)\overline{\psi}(x)\gamma^\mu\psi(x) \tag{I-30}$$

where

$$\mathcal{L}_{em} = -qA_\lambda(x)\ j^\lambda(x) \tag{I-31}$$

is the interaction Langrangian density for a particle of charge q in the potential $A_\mu(x)$, and

$$j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x) \tag{I-32}$$

is the current of the particle. The resemblance between (I-32) and (I-21) is only in appearance since the wave functions $\psi(x)$ and $\overline{\psi}(x)$ no longer satisfy the free equations (I-19) and (I-20). Rather, the equations of motion resulting from (I-30) are

$$(i\gamma^{\mu}\partial_{\mu} - q\gamma^{\mu} A_{\mu}(x) - m)\psi(x) = 0 \qquad \text{(I-33)}$$

$$(i\partial_{\mu}\overline{\psi}(x)\gamma^{\mu} + qA_{\mu}(x)\overline{\psi}(x)\gamma^{\mu} + m\overline{\psi}(x)) = 0 \qquad \text{(I-34)}$$

These equations of motion may be used to show that the current (I-32) for the interacting fields is conserved; and this current may be used as the source in (I-18), with the identification $J^{\mu}(x) = qj^{\mu}(x)$. The resulting set of coupled equations

$$\begin{aligned} (i\gamma^{\mu}\partial_{\mu} - m)\psi(x) &= qA_{\mu}(x)\gamma^{\mu}\psi(x) \\ \square A^{\mu}(x) = q\, j^{\mu}(x) &= q\overline{\psi}(x)\gamma^{\mu}\psi(x) \end{aligned} \qquad \text{(I-35)}$$

form the foundations of QED.

Equations (I-35) are of such mathematical complexity that it has not yet been possible to find closed form solutions for them. There has, however, been great success in the use of successive approximation schemes, or perturbation approximations,[4] which have descriptive physical interpretations.[5]

The complicating feature of equations (I-35) is the self-interaction of the charge. The current j^{μ} produces a potential A^{μ} which in turn affects the current. This self-interaction feature causes divergences in approximations of order higher than the first. Nevertheless, such divergences may be treated in a consistent manner so that the final results are finite.

A distinction can be made between external fields A_μ^{ext} and fields produced by the currents of the particles. For example, the second of equations (I-35) can be written

$$\Box A^\mu(x) = q\ j^\mu(x) + J^\mu(x)^{ext} = q\bar{\psi}(x)\gamma^\mu\psi(x) + J^\mu(x)^{ext}$$

where $A^\mu(x)$ is the total potential created by the particle currents and the external current. However, if $J^\mu(x)^{ext}$ is a current whose source is external to the region of interaction, then $J^\mu(x)^{ext}$ will vanish in the vicinity of the particles, where the source potential will satisfy

$$\Box A^\mu(x)^{ext} = 0$$

The potential $A^\mu(x)^{ext}$, created by the external current, does not vanish near the particles, however, and must be included in the first of equations (I-35).

While the wave mechanics approach can be made to treat QED, the quantized field approach is better suited for processes in which the number of particles is not constant. The fields appearing in quantum field theory are operators which create and annihilate particles at certain positions. In the first approximation of a particle scattering from a potential, for example, one field annihilates the incoming particle and another field creates the outgoing particle; and the interaction with the potential occurs at the point of annihilation and creation. The transition probability amplitude is the integral over position of the amplitude for the transition occurring at a certain point. In the next chapter, explicit expressions for the free fields (sufficient for this work, since only first approximations will be used) are given in terms of free particle creation and annihilation operators.

The fundamental equations of QED, for massive, spin 1/2 particles, in the field theoretic formalism (hereafter, QED will mean the field theoretic formalism of relativistic, quantum electrodynamics) are equations (I-35), with quantum field operators instead of wave functions.

The interaction Hamiltonian density $\mathcal{H}(x)$ is given by

$$\mathcal{H}(x) = q\, j^{\mu}(x)\, A_{\mu}(x) \qquad \text{(I-36)}$$

except when there is an interaction with an external field, in which case the additional term

$$\mathcal{H}_I'(x) = q\, j^{\mu}(x)\, A_{\mu}(x)^{ext}$$

must be included. Standard, detailed calculations (see, for example, the texts by Bjorken and Drell[6] and Gasiorowicz,[7] and references contained in them) using reduction techniques and perturbation theory lead to the Feynman rules for calculation of the S-matrix elements, i.e., the transition probability amplitudes, from which scattering cross sections are obtained. The S-matrix elements may be calculated from the Dyson formula for the S-operator:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \ldots dx_n \; T\{\mathcal{H}(x_1) \ldots \mathcal{H}(x_n)\} \qquad \text{(I-37)}$$

$$= T \left\{ e^{-i \int_{-\infty}^{\infty} dx \mathcal{H}(x)} \right\}$$

The lowest interesting approximation is first-order perturbation theory, in which the S-operator is approximated by

$$S - 1 = -i \int dx \mathcal{H}(x)$$

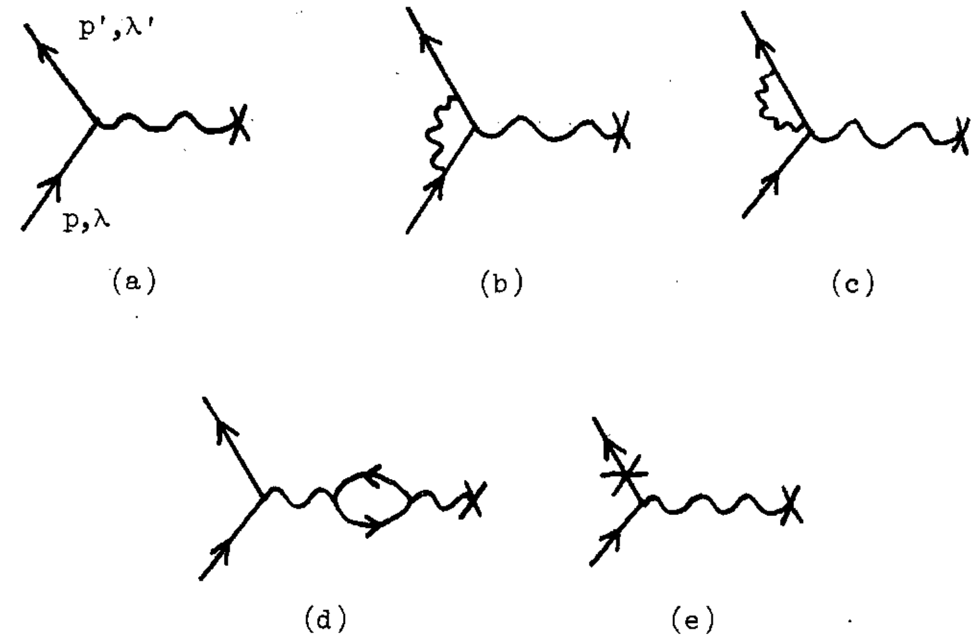
As examples, first-order Coulomb scattering of electrons and lowest order electron scattering from a Dirac proton are discussed. Figure 1(a) shows an electron with initial momentum p and spin projection λ being scattered in first order by an external Coulomb potential into the outgoing state with momentum p' and spin projection λ'. The transition amplitude for the process is

$$\langle p'\lambda'|S-1|p\lambda\rangle = ie \int dx \langle p',\lambda'|j^{\mu}(x)|p\lambda\rangle \, A_{\mu}^{ext}(x)$$

where $j^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x)$ is the electron current. For a fixed point charge Ze, the external potential is given by

$$\delta_{\mu 0} \frac{Ze}{4\pi|\vec{x}|}$$

When the initial electron beam is unpolarized and the polarization of the scattered electrons is not observed, the initial spin states must be averaged and the final spin states summed. The cross section for scattering into the solid angle $d\Omega$ with scattering angle θ is



(a) Lowest Order

(b) and (c) Radiative Corrections

(d) Vacuum Polarization

(e) Mass Renormalization

Figure 1. Coulomb Scattering of Electrons

Figure 2. Electron Scattering From a Dirac Proton

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha m}{2p^2 \sin^2\frac{\theta}{2}}\right)^2 \left(1 + \frac{p^2}{m^2}\cos^2\frac{\theta}{2}\right) \tag{I-38}$$

where energy conservation requires $|\vec{p}\,'| = |\vec{p}| = p$. This differs from the Rutherford formula in the term $(p^2/m^2)\cos^2\theta/2$, which arises from the interaction of the electron magnetic moment with the magnetic field seen by the electron; and this term vanishes in the non-relativistic limit.

If the scattering is from a potential produced by a fixed charge distribution $\rho(\vec{x})$, then the potential is

$$A_\mu^{ext}(\vec{x}) = \delta_{\mu 0}\,\frac{Ze}{4\pi}\int d\vec{x}\,'\,\frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$$

where $\rho(\vec{x})$ is normalized to unity. The scattering cross section in this case is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha m}{2p^2 \sin^2\frac{\theta}{2}}\right)^2 \left(1 + \frac{p^2}{m^2}\cos^2\frac{\theta}{2}\right)|F(\vec{p}\,'-\vec{p})|^2$$

where $F(\vec{p}\,'-\vec{p})$ is the form factor defined by (I-9).

The case of electrons scattering from protons, depicted in Figure 2, is more complicated. Here, in lowest order, the electron scatters from the potential produced by the proton, and vice versa. The transition amplitude is proportional to

$$\int dx_1 dx_2 \{\langle k'\lambda'|-ej_e^\mu(x_1)|k\lambda\rangle\}\{\frac{-g_{\mu\beta}}{2}\,D_F(x_1-x_2)\}\{\langle p'\alpha'|ej_p^\beta(x_2)|p\alpha\rangle\} \tag{I-39}$$

where the function $D_F(x)$ is given by*

$$\frac{D_F(x)}{2} = \frac{i}{(2\pi)^4} \int dk \, \frac{e^{-ikx}}{k^2 + i\varepsilon}$$

The factors in the last two brackets in (I-39), when the x_2 integration is performed, effectively give the potential (produced by the proton current j_p^μ) with which the electron current interacts. Similar considerations relate the first two brackets to the potential with which the proton interacts. To treat the proton as a Dirac proton (a heavy positron) means to use

$$j_p^\mu(x) = \overline{\psi}_p(x)\gamma^\mu\psi_p(x) \qquad \text{(I-40)}$$

as the proton current. The unpolarized cross section for electrons scattering from dirac protons is obtained from (I-39) by use of (I-40). Two cases of interest occur: they are the cases when proton recoil may be neglected (when the electron energy E is much less than the proton rest energy M), and when proton recoil is important. In the latter case, the electron is extremely relativistic, and correction terms of the form $m/E \ll 1$ may be neglected.

For the case in which proton recoil may be neglected, the cross section reduces to (I-38), with Z = 1. When proton recoil is important, the cross section takes the form

*This is the photon propagator of Gasiorowicz[7].

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{\cos^2 \frac{\theta}{2}}{1 + \frac{2E}{M} \sin^2 \frac{\theta}{2}} \left[1 - \frac{t}{2M^2} \tan^2 \frac{\theta}{2} \right] \tag{I-41}$$

where $t = (k'-k)^2 = (p-p')^2$ is the momentum transfer.

Deviations of experimental data from the cross section (I-41) occur when the electron wavelength is on the order of, or smaller than, approximately 5f. Such deviations are interpreted as proton structure, meaning the proton is not a Dirac particle. To defend this interpretation, we must convince ourselves that the electron is a point particle. The validity of QED applied to electrons means that electrons act as point particles, within presently available experimental accuracies and energy limitations of electron accelerators. We now discuss several tests of QED, among them the Lamb shift in hydrogen and the anomalous magnetic moment of the electron; we also briefly mention the results of some high energy colliding beam experiments. The articles by Gatto[8] present good discussions on the tests of QED, while the more recent preprint by Lautrup[2] includes newer theoretical and experimental results. Extensive references are contained in the Gatto articles and the Lautrup preprint.

Classically, the relation between the magnetic moment $\vec{\mu}$ of a particle of charge q and mass m in a circular orbit and the orbital angular momentum of the particle $\vec{L}$ is $\vec{\mu} = \frac{q}{2m} \vec{L}$. This relation is also found to be correct in quantum systems. However, the analogous relation does not hold for the intrinsic magnetic moment of the electron (as well as for many other particles). Rather, the intrinsic magnetic moment of the electron is related to its spin $\vec{S}$ by

$$\vec{\mu} = g(q/2m)\,\vec{S}$$

where g is called the g-factor. The nonrelativistic limit of the Dirac equation, with minimal coupling to the electromagnetic field, predicts the value of the electron g-factor to be $g = 2$. In the late 1940's, however, QED calculations were found to be in agreement with an experimentally observed anomalous addition to the g-factor.

When the scattering of an electron by an external field is considered in higher than first order, the processes variously called radiative corrections, vacuum polarization, and mass renormalization (some of these processes are shown in Figure 1 in second order) become extremely important. In addition, the fact that experimental energy resolution is finite means that elastic scattering cannot be distinguished from scattering in which soft bremsstrahlung production occurs. When soft bremsstrahlung production is taken into account in the calculation of cross sections, it eliminates the problem of infrared divergences which arises because of the zero mass of the photon.

When first-order calculations of the electron interaction with the electromagnetic field are made, the relation $g = 2$ results exactly. However, when higher order calculations are carried through, the value found for g is slightly larger than 2. The contribution to the magnetic moment arising from this addition is the anomalous magnetic moment, and may be described in terms of the parameter

$$a = \frac{g - 2}{2}$$

The result of QED calculations through sixth order for the electron anomaly is[2]

$$a_e^{QED} = 0.5\left(\frac{\alpha}{\pi}\right) - 0.32848 \left(\frac{\alpha}{\pi}\right)^2 + (1.49 \pm 0.20) \left(\frac{\alpha}{\pi}\right)^3 \qquad \text{(I-42)}$$

The experimental result for the electron anomaly is

$$a_e \times 10^9 = 1159657.7 \pm 3.5 \qquad \text{(I-43)}$$

When the fine structure constant, with the non-QED experimental value

$$\alpha^{-1} = 137.03608 \pm 0.00026$$

is used in (I-42), there results

$$a_e^{QED} \times 10^9 = 1159655.4 \pm 3.3$$

Table 1 (taken from Lautrup[2]) shows a comparison of theory and experiment for the electron anomaly; the agreement is excellent.

The Lamb shift in hydrogen refers to the splitting of the $2S_{1/2}$ and $2P_{1/2}$ levels which are degenerate in the Dirac theory. Higher order effects in QED, as well as effects of the finite size and mass of the proton, remove this degeneracy. Both levels are affected, but not equally, and a shift is therefore predicted. With the previously stated value for α^{-1}, the theoretical prediction for the frequency shift is

$$(1057.911 \pm 0.012) \text{ MHz}$$

Table 1. Comparison of Theoretical and Experimental Values of the Anomalous Magnetic Moment of the Electron (from Lautrup[2])

Effect		$a_e \times 10^9$
2nd order QED		1161409.0 $\pm$ 2.2*
4th order QED		-1772.3
6th order QED		18.7 $\pm$ 2.5**
8th order QED (estim.)		-0.04
Hadronic		0.003
Weak		0.00005
Total	theory	1159655.4 $\pm$ 3.3
	experiment	1159657.7 $\pm$ 3.5

*Uncertainty in α
**Uncertainty in integration

while the observed shift is

$$(1057.90 \pm 0.06) \text{ MHz}$$

the two values being in good agreement.

The small distance behavior of QED can be tested with high energy, high momentum transfer experiments, with relatively low precision required, as well as with annihilation experiments. With storage rings, such high energy colliding beam processes as

$$e^- + e^- \rightarrow e^- + e^- \qquad \text{Møller Scattering}$$

$$e^- + e^+ \rightarrow e^- + e^+ \qquad \text{Bhabha Scattering}$$

$$e^- + e^+ \rightarrow \begin{cases} \mu^- + \mu^+ \\ 2\gamma \end{cases} \qquad \text{Annihilation}$$

can be studied. Comparisons of experiment with theory for these processes indicate that QED with a massless photon and a point electron is valid to distances of roughly 0.1f or less.

These examples, along with numerous other data, show no serious discrepancy between QED and experiment; and they indicate within present precision and energy limitations that the electron may be considered a point Dirac particle. That the electron has an anomalous magnetic moment is not due to structure; rather, the electron anomaly is explained by radiative corrections to the scattering of a point Dirac particle.

Electron-Nucleon Scattering

From the previous discussion it may be concluded that the electron is the best available probe for studying nuclear electromagnetic structure. The primary concern in this work is the electromagnetic structure of the nucleons; the results of the work done concerning charge distributions of medium and heavy nuclei are only briefly mentioned here.

In the late 1940's and early 1950's there became available electron accelerators which were capable of producing electrons with wavelengths small enough to begin detecting finite nuclear sizes by electron scattering (see Hofstadter[9] for a collection of reprints concerning electron scattering and nuclear and nucleon structure). With the development of higher energy electron beams, the charge distributions in nuclei were probed in more detail. For comparative purposes, a model for the nuclear charge distribution is assumed, and scattering cross sections are calculated. The calculations for medium and heavy nuclei must be made, for example, using phase shift calculations, since the Born approximation is not very good for high Z nuclei. The nuclear charge models which seem to fit the data best are those of the Fermi distribution type (relatively flat near the origin and dropping off near the "edge"), and a variation which is basically similar to the Fermi distribution, but which has a slight dip in the charge density near the center. All the successful charge density models (those which satisfactorily explain the data) have skin thicknesses of roughly 2.5f rather than sharply defined edges.

Because the nucleons are substantially smaller than heavy nuclei, higher energy electrons are needed to probe details of nucleon structure than are needed for probing heavy nuclei. Around the middle and late 1950's (see Hofstadter[9]), experiments on electron-nucleon scattering began showing some of the details of nucleon structure indicating nucleon "sizes" of about .6 to .8f (rms charge radius of proton and rms magnetic moment radius of neutron, for example). We now discuss electrons scattering from massive, spin 1/2 particles, electron-nucleon scattering being a special case.

The lowest order elastic scattering process in electromagnetic interactions is one photon exchange between the currents of the scattering particles. This process is illustrated for electron-proton scattering in Figure 2. The transition amplitude for this process has already been given by (I-39); and since the forms of the matrix element of the electron current and the photon propagator are known from QED, the object of primary interest becomes the matrix element of the current of the massive, spin 1/2 particle

$$\langle p'\lambda' | j^{\mu}(x) | p\lambda \rangle$$

where $p\lambda$ and $p'\lambda'$ are the initial and final momenta and spin projections of the particle. A translation of the space-time origin results in

$$e^{-i(p-p')x} \langle p'\lambda' | j^{\mu}(0) | p\lambda \rangle$$

and it is the matrix element

$$\langle p'\lambda' | j^{\mu}(0) | p\lambda \rangle \qquad \text{(I-44)}$$

that is of interest. We can use the properties of j^{μ}, which are required in order that it may be identified with the electromagnetic current of the particle, to set limits on the possible forms of expression (I-44). The properties of j^{μ} referred to are due to the requirements that $j^{\mu}(x)$: (1) transforms as a vector under homogeneous Lorentz transformations, (2) transforms as a vector under space inversion, (3) transforms as a pseudovector under time inversion, and (4) is an hermitian operator. There are also the additional requirements that $j^{\mu}(x)$ must be conserved because of charge conservation and must negate itself under charge conjugation. From these required properties of j^{μ}, it follows that the most general possible form of the matrix element (I-44) is

$$\frac{1}{(2\pi)^3}\,\overline{U}^{(\lambda')}(p')\{[F_1(t) + 2MF_2(t)]\gamma^{\mu} - F_2(t)(p^{\mu}+p'^{\mu})\}U^{(\lambda)}(p) \qquad \text{(I-45)}$$

where $t = (p'-p)^2$ is the invariant momentum transfer, F_1 and F_2 are arbitrary invariant functions of t, and the wave function $U^{(\lambda)}(p)$ is given by

$$U^{(\lambda)}(p) = \frac{(p^{\mu}\gamma_{\mu} + M)}{\sqrt{2M(M + \omega)}}\,U^{(\lambda)}(0) \qquad \text{(I-46)}$$

and the definitions

$$U^{(+\ 1/2)}(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad U^{(-\ 1/2)}(0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \text{(I-47)}$$

The use of (I-46) and (I-47) in (I-45) implies spin quantization along the third axis. It should be pointed out that Dirac's equation does not have to be used in arriving at the expression (I-45), so the assumption has not been made that the particles are Dirac particles (see, for example, Kramer[10]).

Expression (I-45) defines the relativistically invariant form factors $F_1(t)$ and $F_2(t)$, called the Dirac and Pauli form factors respectively. It is through these form factors that electromagnetic structure of massive, spin 1/2 particles is defined. A particle is said to have no electromagnetic structure (i.e., to be a point particle) if its form factors satisfy

$$F_1(t) = \text{constant} \qquad F_2(t) = 0$$

Thus, a particle is said to have electromagnetic structure if $F_1(t)$ is not constant and/or $F_2(t)$ is not zero, although F_1 = constant and F_2 = constant may be considered point structure.

It may be noted here that when two particles with structure scatter, the current matrix element for each particle would be an expression like (I-45). Each particle would thus contribute its own form factors.

The nonrelativistic limit of the interaction of the current $j^\mu(x)$ with an external electromagnetic field shows that

$$e\ F_1(0)$$

is the effective electric charge of the particle, while

$$\frac{e}{2M}\ [F_1(0) + 2M\ F_2(0)]$$

is the magnetic moment of the particle, from which it follows that

$$a = \frac{g - 2F_1(0)}{2} = 2M\ F_2(0)$$

is the anomalous contribution to the magnetic moment.

Although there is some difficulty in defining the relation between relativistic form factors and charge and magnetic moment distributions, there is a preferred combination of the Dirac and Pauli form factors defined by

$$\begin{aligned} G_E(t) &= F_1(t) + \frac{t}{2M} F_2(t) \\ G_M(t) &= F_1(t) + 2M\ F_2(t) \end{aligned} \tag{I-48}$$

These Sach's form factors, the electric (G_E) and the magnetic (G_M) form factors, can be interpreted in the Breit frame (the frame in which $\vec{p}\,' = -\vec{p}$) as the Fourier transforms of the charge and magnetic moment distributions. Whatever the interpretation, the form factors satisfy the conditions that $eG_E(0)$ is the charge of the particle and $\frac{e}{2M} G_M(0)$ is its magnetic moment.

The scattering cross section following from (I-39), (I-45), and (I-48) is the Rosenbluth formula

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left.\frac{d\sigma}{d\Omega}\right]_{pt} \left\{ \frac{G_E^{\ 2}(t) - \frac{t}{4M^2} G_M^{\ 2}(t)}{1 - \frac{t}{4M^2}} - \frac{t}{2M^2} G_M^{\ 2}(t)\ \tan^2 \frac{\theta}{2} \right\} \\ \left.\frac{d\sigma}{d\Omega}\right]_{pt} &= \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}}\ \frac{\cos^2 \frac{\theta}{2}}{1 + \frac{2E}{M} \sin^2 \frac{\theta}{2}} \end{aligned} \tag{I-49}$$

where $d\sigma/d\Omega)_{pt}$ is the cross section for electrons scattering from a spinless particle of mass M and charge e. In (I-49) E is the electron energy and θ the electron scattering angle observed in the laboratory. The ratio $d\sigma/d\Omega \big/ d\sigma/d\Omega)_{pt}$, evaluated for constant t, yields a linear function of $\tan^2 \theta/2$, with the slope and intercept giving information about the form factors. This expected linear behavior has been checked experimentally and found to hold. The talk on "The Rosenbluth Formula" by D. Yennie[11] elaborates on the assumptions underlying the Rosenbluth formula, experimental evidence in its favor, and corrections expected from QED which are taken into account when comparisons are made with experiment.

Figures 3 and 4 (from Weber[12]) show the form factors of the proton (G_{EP} and G_{MP}) and neutron (G_{En} and G_{Mn}). In the plots, $q^2 = -t$, and μ_p and μ_n are the proton and neutron magnetic moments. The data shown are consistent with and are summarized by the dipole fit[13]

$$G_{Ep}(t) \cong \frac{G_{Mp}(t)}{\mu_p} \cong \frac{G_{Mn}(t)}{\mu_n} \cong \frac{4M^2}{t} \frac{G_{En}(t)}{\mu_n} \cong (1 - t/0.71)^{-2}$$

Actually, the data neither support nor deny the relation stated for $G_{En}(t)$ because of the large experimental uncertainties. The data are consistent with a small, nonzero neutron charge form factor whose slope at $t = 0$ is indicated by the dashed lines in the plots,[14] the low energy electron-neutron interaction being explained[15] by the neutron magnetic moment.

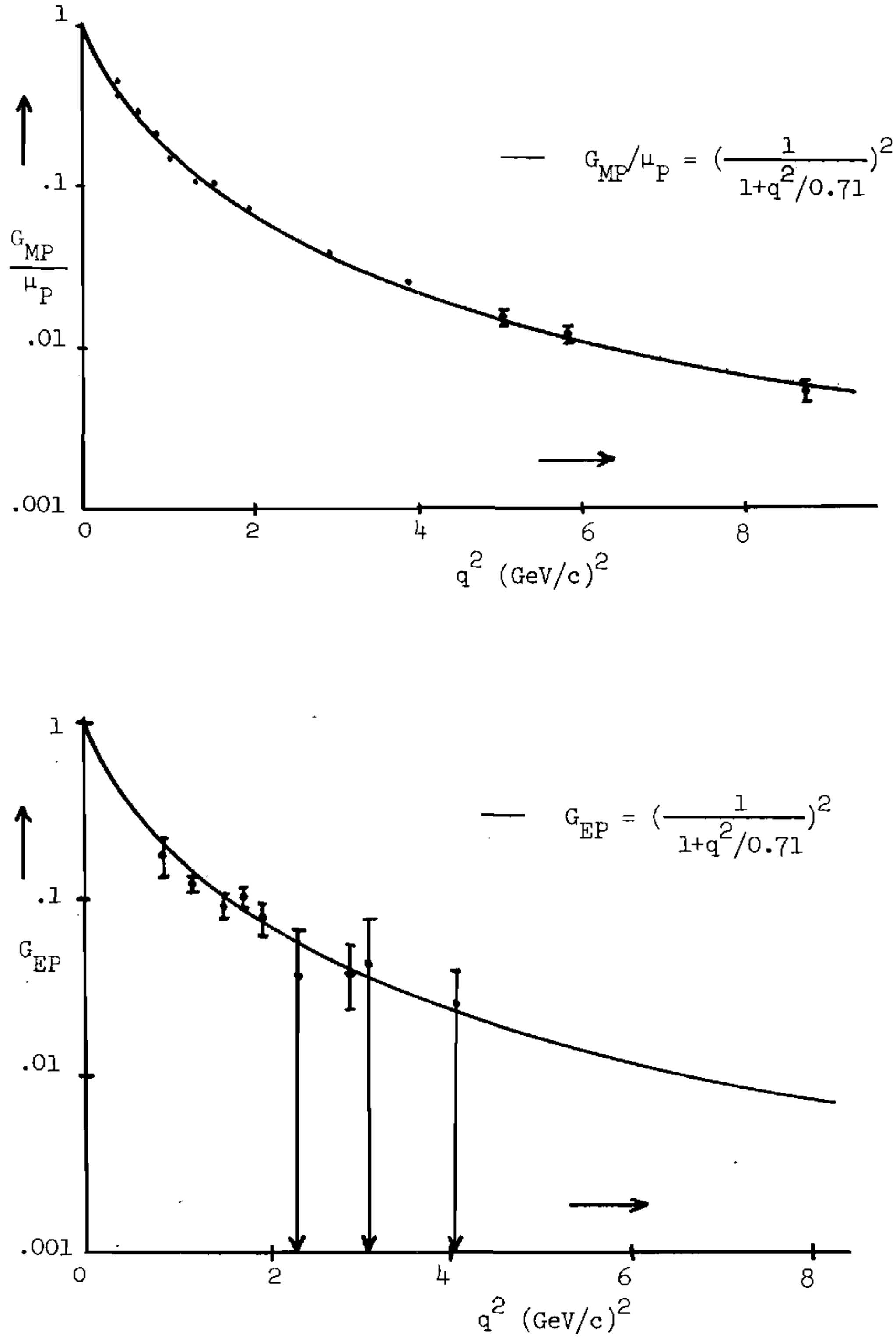


Figure 3. Proton Form Factors (from Weber [12])

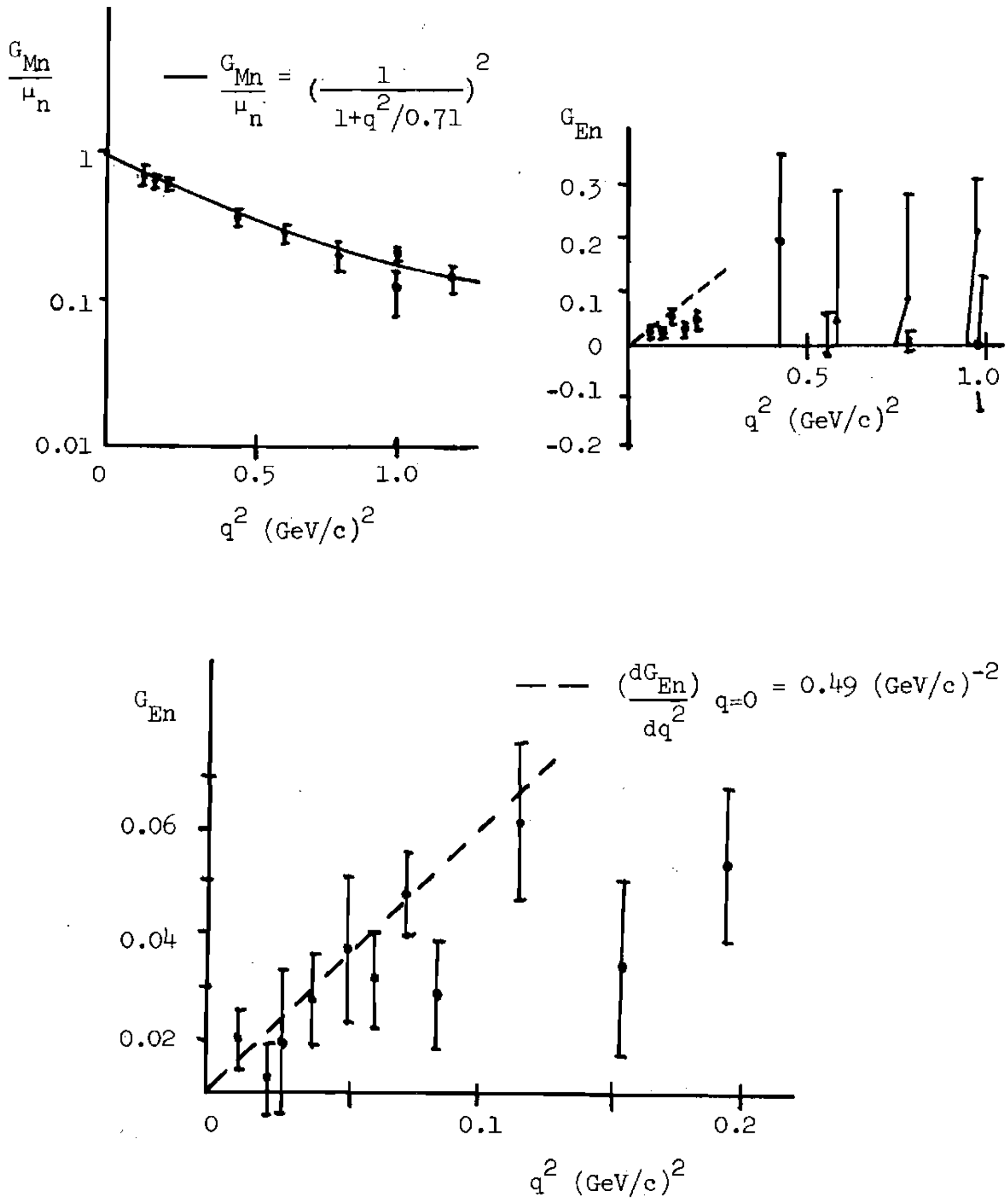


Figure 4. Neutron Form Factors (from Weber [12])

The interpretation of the nucleon form factors originates in the fact that the nucleons undergo strong interactions and may thus be thought to have a "meson cloud" surrounding them. Interactions of photons with such mesons would then appear as electromagnetic structure of the nucleons. This situation is depicted in Figure 5. The strong interaction perturbation approach[16] to the nucleon form factors does indicate the possibility of explaining nucleon structure via pions, but the numbers obtained for the anomalous magnetic moments, for example, are not in good agreement with experiment (see Drell and Zachariasen[17] for a discussion of the perturbation approach).

A more recent and successful approach to nucleon structure has made use of dispersion relations[18,19] arising from the analyticity and unitarity properties of the current matrix elements. (Discussions of dispersion relations in relation to form factors may be found, for example, in the texts by Bernstein,[20] Drell and Zachariasen,[17] and Gasiorowicz.[7]) Since iso-spin is conserved by the strong interactions, it is convenient to write the electromagnetic current of the nucleon as the sum of an isoscalar j_μ^s and an isovector j_μ^v defined in terms of the proton and neutron currents by

$$\begin{aligned} j_\mu^p &= j_\mu^s + j_\mu^v \\ j_\mu^n &= j_\mu^s - j_\mu^v \end{aligned} \qquad \text{(I-50)}$$

Corresponding isoscalar and isovector form factors are defined from these currents. The usefulness of this decomposition lies in the considerations of the meson states intermediate between the photon

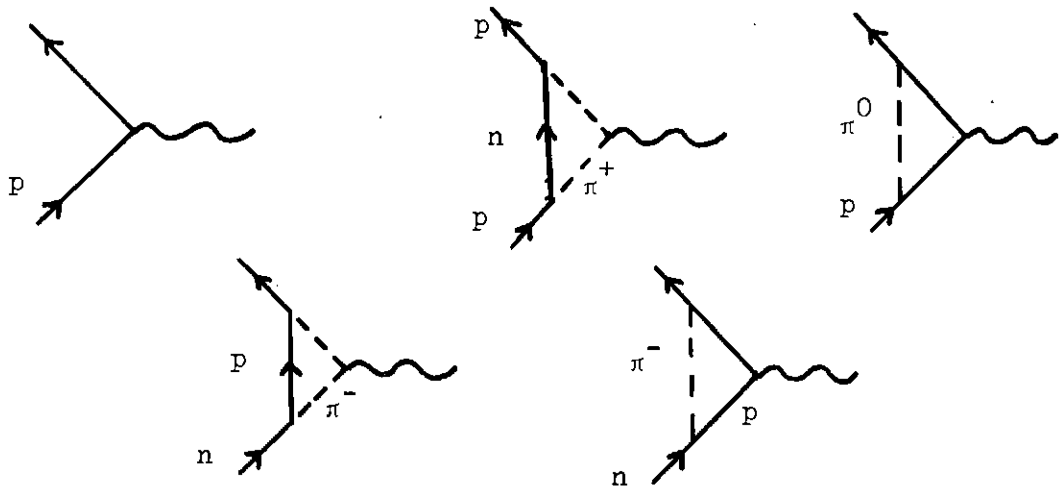


Figure 5. Nucleon Structure Graphs in Strong Interaction Perturbation Approximation

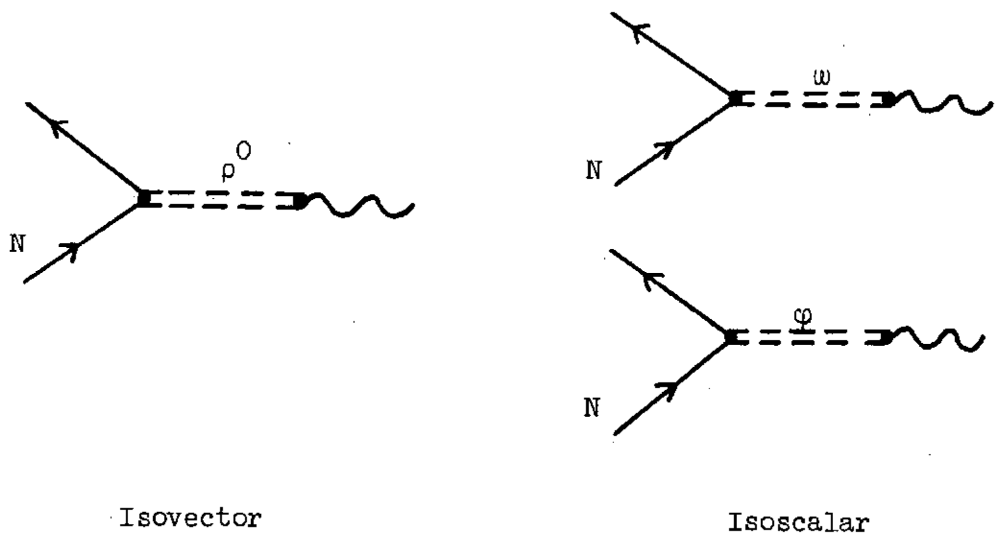


Figure 6. Vector Meson Dominance

and the nucleon. Only isoscalar (isovector) intermediate states contribute to the isoscalar (isovector) form factor. Isoscalar (isovector) pion states which are odd under charge conjugation consist of an odd (even) number of pions. Since the pion is spinless the lowest isoscalar state is a three-pion state, and the lowest isovector state is a two-pion state. Use of such intermediate multi-pion states results in a complicated calculation which is essentially equivalent to the perturbation treatment, and which yields only slightly more encouraging results.[17]

More successful than the previously mentioned approaches has been the idea of vector meson dominance. Here, instead of multi-pion states being intermediate, some of the neutral vector meson resonances are used as states intermediate between the photon and the nucleon. Figure 6 shows graphical representations of the vector meson dominance approximation, with ρ^0 contributing to the isovector form factor and ϕ and ω contributing to the isoscalar form factor. In the approximation of narrow resonances, the expressions found for the form factors have the form[7]

$$G(t) = \sum_{\text{resonances}} \frac{g_r(M_r^2)}{1 - \frac{t}{M_r^2}} + \text{constant}$$

The known vector resonances provide a reasonably good fit to the data (for $-t \lesssim 1.2(\text{GeV}/c)^2$) for the isoscalar form factor but not to the isovector form factor.[21] In order for the resonances to provide a fit to all the data reasonably well at low momentum transfers, either the ρ mass needs to be substantially lower than its fairly well established value, or there needs to exist another isovector resonance with a mass of roughly 1.2GeV. Such a resonance is not known.

Perhaps some combination of the vector meson dominance approximation with the two-pion nonresonant intermediate states might provide a better fit to low -t data, but such considerations are unknown to us. Even a good fit to the low -t data, however, is not enough for an understanding of the nucleon form factors. The fall off like t^{-2} at large momentum transfer, evident from the dipole fit to the data, must also be explained; and the known vector resonances cannot explain such a rapid fall off.

Approaches other than those using resonances and pion intermediate states have been tried (for references, see Drell[22] and Taylor[23]) but have met only with qualified success. All-in-all it must be said that the theory regarding the nucleon form factors, although somewhat encouraging from several angles, is not very good.

Objective

It is not the purpose of this work to put the understanding of nucleon form factors on a firm footing. An understanding of strong interaction dynamics is surely required for that, and this work does not deal with the strong interactions except, perhaps, in some hidden, indirect way. Rather, the objective here is the investigation of the electromagnetic interactions of elementary particles (massive, spin 1/2 particles specifically) by means of their currents. In the context of field theoretic perturbation approximations, the free particle matrix elements of the currents provide the description of the electromagnetic interactions and properties of the particles; and for this reason we are interested in the form factors. Lorentz covariance and symmetry requirements do not unambiguously determine the types of field operators one can use in the construction of electromagnetic currents, and in general different field operators result in different interactions. Such differences appear in the form factors of the particles.

In this work, then, general electromagnetic currents are constructed using field operators which transform according to finite-dimensional representations of the Lorentz group. Field equations are not emphasized. The guiding principle will be simplicity: form the current operator from the basic combination of one destruction field and one creation field. Matrix elements of the current then yield expressions for the form factors of the particles. Only those currents yielding unphysical results or results at odds with experiment can be ruled out as candidates for electromagnetic currents of particles. As we shall see, the Dirac current is the simplest possible current for a massive, spin 1/2 particle allowed by Lorentz covariance and symmetry requirements. The question to which this work is addressed is essentially this: does nature use any of the other allowed fields?

The finite-dimensional, irreducible representations of the quantum mechanical Lorentz group and free fields transforming according to such representations are discussed in Chapter II, which also presents the manner in which field operators are coupled to form a current. Chapter III shows the calculation of the current matrix elements and the resulting form factors, with a discussion following in Chapter IV.

CHAPTER II

FIELDS AND CURRENTS

The purpose of this chapter is the development of a general expression for the electromagnetic current j^μ in terms of creation and annihilation operators. After a discussion of the current and its properties, the representations of the Lorentz group* (or its quantum mechanical equivalent SL(2,c)) are discussed. Next there is a discussion of fields (linear combinations of creation and annihilation operators) which transform according to irreducible representations of SL(2,c). The symmetry properties of the fields are presented, and the fields are coupled together to form an operator which is a Lorentz vector. Finally, the current conservation condition leads to field equations.

Introduction

As is evident from the matrix element (I-44), the current operator must destroy an initial state and create a final state. It therefore needs to be constructed from creation and annihilation operators, whose transformation properties under the Lorentz group are not convenient. For calculational ease therefore, linear combinations of these operators are made so as to have simple Lorentz

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*Unless otherwise stated, Lorentz group means the homogeneous Lorentz group. A brief discussion of the Lorentz group and SL(2,c) (and references to more extensive presentations) is contained in the book by Streater and Wightman.[24]

transformation properties. We use only finite component fields, and they transform according to finite-dimensional representations of the Lorentz group. Such fields can be decomposed into "sub-fields" which each transform according to irreducible representations of the Lorentz group. Thus, our primary interest is in fields which transform according to finite-dimensional, irreducible representations of the Lorentz group. These irreducible fields are coupled (one creation field and one annihilation field) in such a manner as to form a Lorentz vector operator. The vector is then made to have the symmetries and properties required of an electromagnetic current which are listed in the following. (1) Under Lorentz transformations $x^{\mu} \xrightarrow[(\Lambda)]{} x'^{\mu} = \Lambda^{\mu}{}_{\lambda} x^{\lambda}$ which do not contain space or time inversion, $j^{\mu}(x)$ must be a vector, i.e., $j^{\mu}(x)$ must satisfy the transformation

$$j^{\mu}(\Lambda x) = \Lambda^{\mu}{}_{\lambda}\ U(\Lambda)\ j^{\lambda}(x)\ U^{-1}(\Lambda)$$

or

$$U(\Lambda)\ j^{\mu}(x)\ U^{-1}(\Lambda) = \Lambda_{\lambda}{}^{\mu}\ j^{\lambda}(\Lambda x)$$

where $U(\Lambda)$ is the unitary operator which transforms the state $|\psi\rangle$ into the state $|\psi'\rangle = U(\Lambda)|\psi\rangle$ under the Lorentz transformation Λ. (2) Under space inversion $j^{\mu}(x)$ must be a vector. The space inversion transformation Σ is defined by $x = (x^0,\vec{x}) \xrightarrow[(P)]{} \Sigma x = (x^0,-\vec{x})$. The requirement on $j^{\mu}(x)$ is then the transformation

$$P\ j^{\mu}(x)\ P^{-1} = \Sigma_{\nu}{}^{\mu}\ j^{\nu}(\Sigma x) = \bar{j}^{\mu}(\ x)$$

where

$$\bar{j}^{\mu} = (j^0, -\vec{j})$$

(3) The time inversion transformation is $-\Sigma$, and the hermitian operator (requirement 6) j^{μ} transforms as

$$T\, j^{\mu}(x)\, T^{-1} = -(-\Sigma)_{\nu}{}^{\mu}\, j^{\nu}(-\Sigma x) = \bar{j}^{\mu}(-\Sigma x)$$

which is the transformation of a time inversion pseudovector. The time inversion operator T is anti-unitary. (4) The current must be conserved and thus must satisfy the equation

$$\partial_{\mu}\, j^{\mu}(x) = 0$$

(5) Under charge conjugation the current must satisfy

$$C\, j^{\mu}(x)\, C^{-1} = -j^{\mu}(x)$$

(6) The current must be hermitian. These properties are not all independent for local currents constructed from finite fields (PCT theorem for example), and in fact we do not specifically use requirement (5) since normal ordered operators are used. These properties are enough to uniquely specify the manner in which the minimum number of fields must be coupled to form an electromagnetic current.

Representations of The Lorentz Group

The Lorentz group is defined by the group of real matrices Λ which for arbitrary x and y leave the product $x \cdot y = x^{\mu} y^{\lambda} g_{\mu\lambda}$ invariant under the transformation $x^\mu \xrightarrow[(\Lambda)]{} x'^\mu = \Lambda^\mu{}_\lambda x^\lambda$. This invariance requirement implies the condition

$$\Lambda^\mu{}_\alpha \Lambda^\lambda{}_\beta g_{\mu\lambda} = g_{\alpha\beta}$$

which can be written in the matrix form

$$\Lambda^T G\Lambda = G \tag{II-1}$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

From (II-1) follow the requirements

$$(\Lambda^0{}_0)^2 \geq 1 \quad \text{and} \quad (\det \Lambda)^2 = 1$$

which show that the Lorentz group consists of four disjoint pieces:

$$L_+^\uparrow \ , \ L_+^\downarrow \ , \ L_-^\uparrow, \text{ and } L_-^\downarrow$$

where $\pm$ means $\det \Lambda = \pm 1$, $\uparrow$ means $\Lambda^0{}_0 \geq 1$, and $\downarrow$ means $\Lambda^0{}_0 \leq -1$. Only $L_+^\uparrow$ forms a subgroup since it is the only piece which contains the identity 1. Actually the other pieces are related simply to $L_+^\uparrow$. With Σ representing the matrix for space inversion (P), $-\Sigma$ is the matrix for time inversion (T), and -1 is the matrix for the product of space and time inversion (PT). The relations between the four pieces of the Lorentz group are

$$L_-^\uparrow = \Sigma L_+^\uparrow \ , \ L_-^\downarrow = -\Sigma L_+^\uparrow \ , \ L_+^\downarrow = -1 \ L_+^\uparrow$$

and we need therefore only consider the transformations of fields under $L_+^\uparrow$ (hereafter, unless otherwise indicated, Lorentz group will mean the restricted, homogeneous Lorentz group $L_+^\uparrow$) and under the two improper transformations P and T.

Actually, the group of real interest in quantum mechanics is not the Lorentz group as such, but is a group called the quantum mechanical Lorentz group. In quantum mechanics a symmetry operation is one which leaves probabilities and equations of motion invariant. The probability amplitudes (or transition amplitudes) need not be invariant, but rather may be multiplied by a phase factor under a symmetry operation. For example, fermion states with spin quantized along the z-axis are multiplied by -1 under a rotation of angle 2π about the z-axis, while similar boson states are unaffected. The representations of SL(2,c) (SL(2,c), sometimes called the quantum mechanical Lorentz group, is the group of 2x2 unimodular matrices) can accommodate the transformations of fermions and bosons. A homomorphism of SL(2,c) onto the Lorentz group is presented below along with some of the more useful relations, and further discussion of SL(2,c) and its representations is contained in Appendix B.

From the real vector x^μ we form the hermitian matrix

$$\underset{\sim}{x} = x^\mu \sigma_\mu = \begin{pmatrix} x_0 - x_3 & -x_1 + ix_2 \\ -x_1 - ix_2 & x_0 + x_3 \end{pmatrix} \qquad \text{(II-2)}$$

whose determinant is $x \cdot x$. With A an arbitrary element of SL(2,c), we define a transformation

$$x'^{\mu}\, \sigma_{\mu} = \underset{\sim}{x}' = A\, \underset{\sim}{x}\, A^{\dagger} \tag{II-3}$$

The new matrix $\underset{\sim}{x}'$ is hermitian, and since A is unimodular the determinant of $\underset{\sim}{x}'$ is $x' \cdot x' = x \cdot x$. Thus (II-3) represents a Lorentz transformation:

$$x'^{\mu} = \Lambda(A)^{\mu}{}_{\lambda}\, x^{\lambda}$$

where

$$\Lambda(A)^{\mu}{}_{\lambda} = \frac{1}{2}\, \mathrm{tr}(\bar{\sigma}^{\mu}\, A\, \sigma_{\lambda}\, A^{\dagger}) \tag{II-4}$$

In (II-4) $\bar{\sigma}^{\mu}$ is defined as $(\bar{\sigma}^{\mu}) = (\sigma^0, -\vec{\sigma}) = (\sigma_{\mu})$. From (II-4) follows the important relation (see (B-1) and (B-2))

$$A\, \sigma^{\mu}\, A^{\dagger} = \Lambda(A)_{\lambda}{}^{\mu}\, \sigma^{\lambda} \tag{II-5}$$

A useful relation for an arbitrary element A of SL(2,c) is

$$\sigma^2\, A\, \sigma^2 = A^{-1^T} \tag{II-6}$$

with the analogous relation for σ^{μ} being

$$\sigma^2\, \sigma^{\mu}\, \sigma^2 = \bar{\sigma}^{\mu^T} \tag{II-7}$$

From (II-6) and (II-7) follows a form of (II-5) which is of importance later:

$$A^{-1^T}(C^{1/2}\, \sigma^{\mu})A^{\dagger} = \Lambda_{\lambda}{}^{\mu}\, (C^{1/2}\, \sigma^{\lambda}) \tag{II-8}$$

where the matrix $C^{1/2}$ is defined as

$$C^{1/2} = -i\sigma^2 \tag{II-9}$$

The finite-dimensional representations of SL(2,c) can be decomposed into a direct sum of irreducible representations, and we now present the finite-dimensional, irreducible representations of SL(2,c) along with some discussion of their properties. (A somewhat more detailed presentation is included in Appendix B along with some special relations satisfied by the representation matrices and the Clebsch-Gordan coefficients.)

The matrices $D^j(A)$ defined in (B-5) are irreducible representations of SL(2,c), and the representations $D^j(A)$ and $D^j(A)^{-1\dagger}$ are inequivalent. The finite-dimensional, irreducible representations of SL(2,c) (as given by (B-9)) are

$$D^{A,B}(S)_{ab,a'b'} = D^A(S)_{aa'}\, D^B(S)^{-1\dagger}_{bb'} \tag{II-10}$$

where S is the SL(2,c) element. (Although we shall commonly use Lorentz transformations as the arguments of the matrices, it is to be understood that the argument is really the SL(2,c) matrix which induces the transformation.) These matrices are nonunitary, except for the trivial A = B = 0 case. For rotations these matrices are unitary, but they are then reducible.

With the help of (B-13) and (B-15) we can show the matrix

$$\Sigma^{A_1B_1,A_2B_2:\mu}_{a_1b_1,a_2b_2} = \langle A_1a_1A_2a_2|\tfrac{1}{2}\alpha\rangle(c\sigma^\mu)_{\alpha\beta}\langle\tfrac{1}{2}\beta|B_1b_1B_2b_2\rangle \tag{II-11}$$

satisfies the relation

$$D^{A_1B_1}(\Lambda)^{-1^T}_{a_1b_1,a_1'b_1'}\ \Sigma^{A_1B_1,A_2B_2:\mu}_{a_1'b_1',a_2'b_2'}\ D^{A_2B_2}(\Lambda)^{-1}_{a_2'b_2',a_2b_2}$$

$$= \Lambda_\nu{}^\mu\ \Sigma^{A_1B_1,A_2B_2:\nu}_{a_1b_1,a_2b_2} \tag{II-12}$$

Relation (II-12) and definition (II-11) are important for establishing equations satisfied by irreducible fields and for determining how to form a vector operator from these fields.

This completes our treatment of the representations of the quantum mechanical Lorentz group SL(2,c). Our task now becomes that of finding free fields which transform according to these representations, i.e., irreducible fields.

Irreducible Fields

For the rest state of a particle of mass $m(\neq 0)$, spin j, and z-component of spin σ we use

$$|\vec{0},\sigma>$$

where $\vec{0}$ means the momentum $\vec{p} = 0$, and the labels m and j are understood. These states transform under rotations according to

$$U(R)|\vec{0},\sigma> = |\vec{0},\sigma'> D^j(R)_{\sigma'\sigma}$$

(As with the D-matrices, the arguments of the unitary operators $U(\Lambda)$ are understood to be the SL(2,c) elements inducing the Lorentz transformation Λ.) States of motion are defined by

$$|\vec{p},\sigma\rangle = U(L(\vec{p}))|\vec{0},\sigma\rangle \qquad \text{(II-13)}$$

and we use the invariant normalization

$$\langle\vec{p}\,',\sigma'|\vec{p},\sigma\rangle = 2\omega(\vec{p})\ \delta(\vec{p}-\vec{p}\,')\ \delta_{\sigma\sigma'} \qquad \text{(II-14)}$$

The transformation $L(\vec{p})$ in (II-13) is any Lorentz transformation for which, with $p_R = (m,\vec{0})$,

$$L(\vec{p})^{\mu}{}_{\nu}\, p_R{}^{\nu} = p^{\mu} = (\omega,\vec{p})$$

and the physical significance of the polarization index σ on the left in (II-13) depends on the choice of $L(\vec{p})$. The two commonly used choices for $L(\vec{p})$ are those that generate states of motion which are helicity eigenstates and those which generate states of motion which are called J_z-eigenstates. The helicity states are produced by the choice

$$L(\vec{p}) = R(\hat{p})\ B_3(|\vec{p}|)$$

while the J_z-eigenstates are produced by the choice

$$L(\vec{p}) = B(\vec{p}) = R(\hat{p})\ B_3(|\vec{p}|)\ R^{-1}(\hat{p})$$

with $B_3(|\vec{p}|)$ and $R(\hat{p})$ defined in Appendix B. We use the hermitian $B(\vec{p})$ primarily, but do not yet specialize to that case.

The Lorentz transformation properties of the states (III-13) are determined by

$$U(\Lambda)|\vec{p},\sigma\rangle = U(\Lambda)U(L(\vec{p}))|0,\sigma\rangle = U(L(\Lambda\vec{p}))U(L^{-1}(\vec{\Lambda p})\Lambda L(\vec{p}))|0,\sigma\rangle$$

The transformation $L^{-1}(\vec{\Lambda p})\Lambda L(\vec{p})$ leaves p_R invariant and must therefore be a rotation; it is called the Wigner rotation

$$R(\Lambda,\vec{p}) = L^{-1}(\vec{\Lambda p})\Lambda L(\vec{p}) \tag{II-15}$$

Thus the transformation of $|\vec{p},\sigma\rangle$ is given by

$$U(\Lambda)|\vec{p},\sigma\rangle = |\vec{\Lambda p},\sigma'\rangle\, D^j(R(\Lambda,\vec{p}))_{\sigma'\sigma}$$

In terms of the creation operators for these states, the transformation may be written as

$$U(\Lambda)a^\dagger(\vec{p},\sigma)U^\dagger(\Lambda) = a^\dagger(\vec{\Lambda p},\sigma')D^j(R(\Lambda,\vec{p}))_{\sigma'\sigma} \tag{II-16}$$

The antiparticle creation operator $b^\dagger(\vec{p},\sigma)$ satisfies the same transformation, and the transformation properties of the annihilation operators $a(\vec{p},\sigma)$ and $b(\vec{p},\sigma)$ are found by taking the adjoint of (II-16) and using the unitary of the rotation D-matrices:

$$U(\Lambda)a(\vec{p},\sigma)U^\dagger(\Lambda) = D^j(R^{-1}(\Lambda,\vec{p}))_{\sigma\sigma'}a(\vec{\Lambda p},\sigma) \tag{11-17}$$

Multiplication of (II-16) with ${C^j}^{-1}$ (see (B-6), (B-7), and (B-8)) results in

$$U(\Lambda){C^j}^{-1}_{\sigma\sigma'}a^\dagger(\vec{p},\sigma')U^\dagger(\Lambda) = D^j(R^{-1}(\Lambda,\vec{p}))_{\sigma\sigma'}{C^j}^{-1}_{\sigma'\beta}a^\dagger(\vec{\Lambda p},\beta)$$

(II-18)

Thus the operators $a(\vec{p},\sigma)$, $b(\vec{p},\sigma)$, $C^{-1}{}_{\sigma\sigma'}a^{\dagger}(\vec{p},\sigma')$, and $C^{-1}{}_{\sigma\sigma'}b^{\dagger}(\vec{p},\sigma')$ all have the same Lorentz transformation properties and can be combined to make fields. (The dimensionality of a C-matrix should be understood to be the same as that of the object with which it is contracted.)

Before constructing the fields, we state the transformation properties of the free particle states (via annihilation operators) under space inversion (P), time reversal (T), and charge conjugation (C) (see Weinberg[26]):

$$
\begin{aligned}
P\ a(\vec{p},\sigma)P^{-1} &= \eta_P a(-\vec{p},\sigma) \\
T\ a(\vec{p},\sigma)T^{-1} &= \eta_T C_{\sigma\sigma'} a(-\vec{p},\sigma') \\
C\ a(\vec{p},\sigma)C^{-1} &= \eta_C b(\vec{p},\sigma)
\end{aligned}
\tag{II-19}
$$

The antiparticle transformations are the same, except the antiparticle phase factors are different and are labeled $\bar{\eta}_P$, $\bar{\eta}_T$, and $\bar{\eta}_C$. In (II-19) the operators are J_z-basis operators, i.e., the states of motion are generated by the special boosts $B(\vec{p})$. Also, for translations $x_0 \to x_0' = x_0 + x$ the states transform according to

$$
\begin{aligned}
U(x)\ a(\vec{p},\sigma)\ U^{-1}(x) &= e^{-ip\cdot x}\ a(\vec{p},\sigma) \\
U(x)\ a^{\dagger}(\vec{p},\sigma)\ U^{-1}(x) &= e^{ip\cdot x}\ a^{\dagger}(\vec{p},\sigma)
\end{aligned}
\tag{II-20}
$$

By an irreducible field is meant a linear combination of creation and annihilation operators which transform under the inhomogeneous Lorentz transformation $x \to x' = \Lambda x+a$ according to

$$U(a,\Lambda)\ \psi^{AB}(x)_{ab}\ U^{-1}(a,\Lambda) = D^{AB}(\Lambda^{-1})_{ab,a'b'}\ \psi^{AB}(\Lambda x+a)_{a'b'} \tag{II-21}$$

We start the construction of the fields by use of a Lorentz invariant Fourier transform of a combination of the operators $a(\vec{p},\sigma)$:

$$\psi^{AB}(x)^{(+)}_{ab} = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{2\omega(\vec{p})}\ M^{AB}_{(\vec{p})}{}_{ab,\sigma}\ a(\vec{p},\sigma)e^{-ip\cdot x} \tag{II-22}$$

The (+) in (II-22) means positive frequency field. Equation (II-20) allows (II-22) to be written as

$$\psi^{AB}(x)^{(+)}_{ab} = U(x)\ \psi^{AB}(0)^{(+)}_{ab}\ U^{-1}(x) \tag{II-23}$$

and we need only consider

$$\psi^{AB}(0)^{(+)}_{ab} = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{2\omega(\vec{p})} M^{AB}_{(\vec{p})}{}_{ab,\sigma}\ a(\vec{p},\sigma) \tag{II-24}$$

Application of an arbitrary transformation Λ to (II-24), by use of (II-17) and (II-21), results in a requirement on the function $M^{AB}(\vec{p})_{ab,\sigma}$:

$$D^{AB}(\Lambda^{-1})_{ab,a'b'}\ M^{AB}(\vec{\Lambda p})_{a'b',\sigma'} = M^{AB}(\vec{p})_{ab,\sigma}\ D^{j}(R^{-1}(\Lambda,\vec{p}))_{\sigma\sigma'} \tag{II-25}$$

In the derivation of (II-25) use has been made of the Lorentz invariance of $d\vec{p}/2\omega(\vec{p})$:

$$\frac{d\vec{p}}{2\omega(\vec{p})} = \frac{d(\overrightarrow{\Lambda p})}{2\omega(\overrightarrow{\Lambda p})}$$

If Λ is chosen to be $L^{-1}(\vec{p})$ in (II-25), there results

$$M^{AB}(\vec{p})_{ab,\sigma} = D^{AB}(L(\vec{p}))_{ab,a'b'} \; M^{AB}(0)_{a'b',\sigma}$$

which allows (II-25) to be written as

$$D^{AB}(R^{-1}(\Lambda,\vec{p}))_{ab,a'b'} \; M^{AB}(0)_{a'b',\sigma'} = M^{AB}(0)_{ab,\sigma} \; D^{j}(R^{-1}(\Lambda,\vec{p}))_{\sigma\sigma'} \tag{II-26}$$

As Λ varies over $L_{+}^{\uparrow}$, $R^{-1}(\Lambda,\vec{p})$ varies over all rotations R, and (II-26) can be written as

$$D^{AB}(R)_{ab,a'b'} \; M^{AB}(0)_{a'b',\sigma'} = M^{AB}(0)_{ab,\sigma} D^{j}(R)_{\sigma\sigma'} \tag{II-27}$$

for all rotations R. With the matrix X defined by

$$X_{\alpha\beta} = \langle K\alpha|Aa'Bb'\rangle \; M^{AB}(0)_{a'b',\beta} \tag{II-28}$$

relation (B-13) for the Clebsch-Gordan coefficients allows us to show

$$D^{K}(R)_{\alpha\mu} \; X_{\mu\beta} = X_{\alpha\mu} \; D^{j}(R)_{\mu\beta}$$

for all rotations R, and Schur's lemma tells us

$$X_{\alpha\beta} = \lambda\ \delta_{Kj}\ \delta_{\alpha\beta}$$

where λ is a constant. Finally, using (B-12) we recover $M^{AB}(0)_{ab,\sigma}$ in the form

$$M^{AB}(0)_{ab,\sigma} = \lambda \langle AaBb|j\sigma\rangle$$

and we choose $\lambda = 1$, obtaining

$$M^{AB}(\vec{p})_{ab,\sigma} = D^{AB}(L(\vec{p}))_{ab,a'b'} \langle Aa'Bb'|j\sigma\rangle \tag{II-29}$$

Equations (II-24) and (II-29) used in (II-23) result in

$$\psi^{AB}(x)^{(+)}_{ab} = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{2\omega(\vec{p})} D^{AB}(L(\vec{p}))_{ab,a'b'}\langle Aa'Bb'|j\sigma\rangle a(\vec{p},\sigma)e^{-ip\cdot x} \tag{II-30}$$

Similar arguments lead to the same results for $\psi^{AB}(x)^{(-)}_{ab}$ in which $a(\vec{p},\sigma)e^{-ip\cdot x}$ is replaced by $C^{-1}{}_{\sigma\sigma'}\ b^{\dagger}(\vec{p},\sigma')e^{+ip\cdot x}$, for $\overline{\psi}^{AB}(x)^{(+)}_{ab}$ in which $a(\vec{p},\sigma)e^{-ip\cdot x}$ is replaced by $b(\vec{p},\sigma)e^{-ip\cdot x}$, and for $\overline{\psi}^{AB}(x)^{(-)}_{ab}$ in which $a(\vec{p},\sigma)e^{-ip\cdot x}$ is replaced by $C^{-1}{}_{\sigma\sigma'}a^{\dagger}(\vec{p},\sigma')e^{+ip\cdot x}$. Requirements of microcausality and Lorentz covariant propagators (see Weinberg[26,27]) lead to the following combinations of the $\psi^{(\pm)}$ for particle fields ψ and antiparticle fields $\overline{\psi}$:

$$\begin{aligned} \psi^{AB}(x)_{ab} &= \psi^{AB}(x)^{(+)}_{ab} + (-)^{2A}\ \psi^{AB}(x)^{(-)}_{ab} \\ \overline{\psi}^{AB}(x)_{ab} &= \overline{\psi}^{AB}(x)^{(+)}_{ab} + (-)^{2A}\ \overline{\psi}^{AB}(x)^{(-)}_{ab} \end{aligned} \tag{II-31}$$

where $\psi^{AB}(x)_{ab}$ is specifically given by

$$\frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{2\omega(\vec{p})} D^{AB}(L(\vec{p}))_{ab,a'b'} \langle Aa'Bb'|j\lambda\rangle$$
$$\left\{a(\vec{p},\lambda)e^{-ip\cdot x} + (-)^{2A} C^{-1}{}_{\lambda\lambda'} b^{\dagger}(\vec{p},\lambda')e^{ip\cdot x}\right\} \tag{II-32}$$

and the antiparticle field $\overline{\psi}^{AB}$ is obtained from (II-32) by the replacement of a with b and $b^{\dagger}$ with $a^{\dagger}$. The requirements leading to (II-31) also lead to the relation between spin and statistics

$$a(\vec{p},\lambda)\ a^{\dagger}(\vec{p}',\lambda') - (-)^{2j}\ a^{\dagger}(\vec{p}',\lambda')a(\vec{p},\lambda) = 2\omega(\vec{p})\delta_{\lambda\lambda'}\delta(\vec{p}-\vec{p}') \tag{II-33}$$

Thus, boson operators obey commutation relations while fermion operators obey anticommutation relations.

The antiparticle fields $\overline{\psi}$ are introduced for convenience. The fields ψ^{AB} and their hermitian adjoints $\psi^{AB\dagger}$ do not have the same transformation properties, whereas ψ^{AB} and $\overline{\psi}^{AB}$ transform identically under $L_{+}^{\uparrow}$. However, as direct calculation shows, $\overline{\psi}^{AB}$ and $\psi^{AB\dagger}$ are related by

$$\psi^{AB}(x)^{\dagger}_{ab} = (-)^{A-B+j}\ C^{BA^{-1}}_{ba,b'a'}\ \overline{\psi}^{BA}(x)_{b'a'}$$
$$\overline{\psi}^{AB}(x)^{\dagger}_{ab} = (-)^{A-B+j}\ C^{BA^{-1}}_{ba,b'a'}\ \psi^{BA}(x)_{b'a'} \tag{II-34}$$

where the matrix $C^{AB}_{ab,a'b'}$ is defined as

$$C^{AB}{}_{ab,a'b'} = C^{A}{}_{aa'}\, C^{B}{}_{bb'}$$

To find the discrete transformations of the fields, we apply P, T, and C to (II-34), with (II-19) and the analogous relations for antiparticle operators being used. The phase factors η and $\bar{\eta}$ are restricted by the requirements that the fields ψ^{AB} and $\bar{\psi}^{AB}$ have definite discrete transformations; i.e., the positive and negative frequency parts must transform the same way. This places restrictions on the phase factors in the following way:

$$\bar{\eta}_T = \eta_T^* \qquad \bar{\eta}_C = \eta_C^* \qquad \bar{\eta}_P = (-)^{2j}\eta_P^*$$

With these phase factor relations, the discrete transformation properties of the fields are

$$\begin{aligned}
P\,\psi^{AB}(x)_{ab}P^{-1} &= \eta_P(-)^{A+B-j}\,\psi^{BA}(\Sigma x)_{ba}\\
P\,\bar{\psi}^{AB}(x)_{ab}P^{-1} &= \eta_P^*(-)^{A+B+j}\,\bar{\psi}^{BA}(\Sigma x)_{ba}\\
T\,\psi^{AB}(x)_{ab}T^{-1} &= \eta_T\, C^{AB}{}^{-1}_{ab,a'b'}\,\psi^{AB}(-\Sigma x)_{a'b'}\\
T\,\bar{\psi}^{AB}(x)_{ab}T^{-1} &= \eta_T^*\, C^{AB}{}^{-1}_{ab,a'b'}\,\bar{\psi}^{AB}(-\Sigma x)_{a'b'} \qquad \text{(II-35)}\\
C\,\psi^{AB}(x)_{ab}C^{-1} &= \eta_C\,\bar{\psi}^{AB}(x)_{ab}\\
C\,\bar{\psi}^{AB}(x)_{ab}C^{-1} &= \eta_C^*\,\psi^{AB}(x)_{ab}
\end{aligned}$$

At this point we could develop the field equations satisfied by these fields. However, since the development of such equations does not follow logically from the previous work, we delay the derivation of field equations until after the current has been established. The field equations arise in a natural manner when current conservation is checked.

Currents

It is now our task to find a vector operator suitable as a candidate for an electromagnetic current. We do this in the simplest manner by using one irreducible creation field coupled to one irreducible annihilation field to form a vector. The fields used are arbitrary to within the limitations imposed by the Clebsch-Gordan coefficients; i.e., in $\psi^{AB}(x)_{ab}$ the A and B are adjustable subject only to the triangle inequality $|A-B| \leqq j \leqq A+B$. Of course the requirement that the two fields couple to form a vector is in itself a restriction, but such a restriction arises in fact on the coupling matrix, which we now investigate.

We couple two fields so as to form a vector

$$g^{\mu}(x) = \overline{\psi}^{A_1B_1}(x)_{a_1b_1} M^{A_1B_1,A_2B_2:\mu}_{a_1b_1,a_2b_2} \psi^{A_2B_2}(x)_{a_2b_2} \qquad \text{(II-36)}$$

whose transformation under $L_+^{\uparrow}$ is required to be

$$U(\Lambda)\; g^{\mu}(x)\; U^{-1}(\Lambda) = \Lambda_{\lambda}{}^{\mu}\; g^{\lambda}(\Lambda x) \qquad \text{(II-37)}$$

When (II-21) and (II-36) are used in (II-37) there results the following requirement on $M^{A_1B_1,A_2B_2;\mu}_{a_1b_1,a_2b_2} = M^{(1,2)\mu}$:

$$D^{(1)}(\Lambda)^{-1^T} M^{(1,2)\mu} D^{(2)}(\Lambda)^{-1} = \Lambda_\nu{}^\mu M^{(1,2)\nu} \qquad \text{(II-38)}$$

In (II-38) the abbreviated notation $D^{(1)}(\Lambda) = D^{A_1B_1}(\Lambda)_{a_1b_1,a_1'b_1'}$ has been used. Requirement (II-38) is enough to determine the matrix $M^{(1,2)\mu}$ up to an arbitrary multiplicative constant. To see this we consider the matrix

$$X^{(1,3)} = \Sigma^{(1,2)}{}_\mu \, C^{(2)} \, M^{(2,3)\mu} \, C^{(3)} \qquad \text{(II-39)}$$

where $\Sigma^{(1,2)}{}_\mu = \Sigma^{(1,2)\lambda} g_{\lambda\mu}$ is defined in (II-11), and $C^{(2)} = C^{A_2B_2}_{a_2b_2,a_2'b_2'}$, etc. Relations (II-12) and (II-38), when used with (II-39), result in

$$D^{(1)}(\Lambda)^T \, X^{(1,3)} = X^{(1,3)} \, D^{(3)}(\Lambda)^T$$

and the replacement of Λ with Λ^T results in

$$D^{(1)}(\Lambda) \, X^{(1,3)} = X^{(1,3)} \, D^{(3)}(\Lambda)$$

As this must be true for all $A(\Lambda)$ in SL(2,c), Schur's lemma tells us

$$X^{(1,3)}_{a_1b_1,a_3b_3} = \lambda \, \delta_{(1)(3)} \, \delta_{a_1a_3} \, \delta_{b_1b_3} \qquad \text{(II-40)}$$

where λ is a constant. We use (II-40) and (II-11) in (II-39) to obtain

$$\langle A_1 a_1 A_2 a_2 | \tfrac{1}{2}\alpha\rangle \, (c\sigma_\mu)_{\alpha\beta} \, \langle \tfrac{1}{2}\beta | B_1 b_1 B_2 b_2\rangle \, C^{A_2B_2}_{a_2b_2,a_2'b_2'} \, M^{A_2B_2,A_3B_3:\mu}_{a_2'b_2',a_3'b_3'}$$

$$= \lambda \, \delta_{A_1A_3} \, \delta_{B_1B_3} \, \delta_{a_1a_3} \, \delta_{b_1b_3} \, C^{A_3B_3-1}_{a_3b_3,a_3'b_3'} \tag{II-41}$$

The relation

$$\langle A_1 a_1 A_2 a_2 | A_3 a_3\rangle = (-)^{2A_2} \frac{[A_3]^{1/2}}{[A_1]^{1/2}} \langle A_1 a_1 | A_2 a_2' A_3 a_3\rangle \, C^{A_2}_{a_2'a_2} \tag{II-42}$$

where $[j] = 2j+1$, allows us to write (II-41) as

$$\frac{[1/2]}{[A_1]^{1/2}[B_1]^{1/2}} \langle A_1 a_1 | A_2 a_2' \tfrac{1}{2}\alpha\rangle \, (c\sigma_\mu)_{\alpha\beta} \, \langle B_1 b_1 | B_2 b_2' \tfrac{1}{2}\beta\rangle \, M^{A_2B_2,A_3B_3:\mu}_{a_2'b_2',a_3'b_3'}$$

$$= \lambda \, \delta_{A_1A_3} \, \delta_{B_1B_3} \, \delta_{a_1a_3} \, \delta_{b_1b_3} \, C^{A_3B_3-1}_{a_3b_3,a_3'b_3'}$$

which can be solved by means of (B-12) to obtain

$$(c\sigma_\mu)_{\alpha'\beta'} \, M^{A_2B_2,A_3B_3:\mu}_{a_2b_2,a_3'b_3'} = \lambda \, \frac{[A_3]^{1/2}[B_3]^{1/2}}{[1/2]}$$

$$\langle A_2 a_2 \tfrac{1}{2}\alpha' | A_3 a_3\rangle \, \langle B_2 b_2 \tfrac{1}{2}\beta' | B_3 b_3\rangle \, C^{A_3B_3-1}_{a_3b_3,a_3'b_3'} \tag{II-43}$$

We now multiply (II-43) by $(\vec{\sigma}^\nu \, C^{-1})_{\beta'\alpha'}$, sum on β' and α' using (B-1), and use the relations

$$\langle A_2 a_2 A_1 a_1 | A_3 a_3 \rangle = \frac{[A_3]^{1/2}}{[A_1]^{1/2}} C^{A_1}_{a_1 a_1'} \langle A_1 a_1' | A_3 a_3' A_2 a_2 \rangle C^{A_3 - 1}_{a_3' a_3}$$

and (II-7) to obtain

$$M^{(2,3)\mu} = \left\{ \lambda \frac{[A_3][B_3]}{[1/2][1/2]} (-)^{A_2 + B_2 - A_3 - B_3 - 1} \right\} \Sigma^{(2,3)\mu}$$

We choose

$$\lambda = \omega \frac{[1/2][1/2]}{[A_3][B_3]} (-)^{-A_2 - B_2 + A_3 + B_3 + 1}$$

where ω is a constant, to finally obtain

$$M^{A_1 B_1, A_2 B_2 : \mu}_{a_1 b_1, a_2 b_2} = \omega \Sigma^{A_1 B_1, A_2 B_2 : \mu}_{a_1 b_1, a_2 b_2} \tag{II-44}$$

We now have the Lorentz vector

$$g^{(1,2)\mu}(x) = \omega \bar{\psi}^{A_1 B_1}(x)_{a_1 b_1} \langle A_1 a_1 A_2 a_2 | \tfrac{1}{2}\alpha \rangle (c\sigma^\mu)_{\alpha\beta}$$

$$\langle \tfrac{1}{2}\beta | B_1 b_1 B_2 b_2 \rangle \psi^{A_2 B_2}(x)_{a_2 b_2} \tag{II-45}$$

The ω is an unknown coupling constant to be determined by comparison with experiment, and it turns out to be related to the charge of the particle being described. To form a vector suitable as a candidate for an electromagnetic current we first add to (II-45) its hermitian conjugate:

$$h^{(1,2)\mu}(x) = h^{(1,2)\mu\dagger}(x) = g^{(1,2)\mu}(x) + g^{(1,2)\mu\dagger}(x)$$

With the help of (II-34) we find

$$g^{(1,2)\mu\dagger}(x) = (-)^{2j+1} \frac{\omega^*}{\omega} g^{(\overline{2},\overline{1})\mu}(x)$$

where if (1) stands for the (A_1,B_1) representation labels, $(\overline{1})$ stands for the (B_1,A_1) labels. Thus we have

$$h^{(1,2)\mu}(x) = \omega\overline{\psi}^{(1)}\ \Sigma^{(1,2)\mu}\ \psi^{(2)} + (-)^{2j+1}\ \omega^*\ \overline{\psi}^{(\overline{2})}\ \Sigma^{(\overline{2},\overline{1})\mu}\ \psi^{(\overline{1})}$$

To ensure the correct P transformation properties we form

$$i^{(1,2)\mu}(x) = h^{(1,2)\mu}(x) + P\ \overline{h}^{(1,2)\mu}(\Sigma x)\ P^{-1}$$

where $\overline{h}^{\mu} = h_{\mu}$ (actually $\overline{h}^{\mu}$ means use $\overline{\sigma}^{\mu}$ instead of σ^{μ}). By use of (II-35) we can show

$$P\ \overline{g}^{(1,2)\mu}(\Sigma x)\ P^{-1} = (-)^{A_1+B_1+A_2+B_2+1}\ g^{(\overline{1},\overline{2})\mu}(x)$$

resulting in

$$i^{(1,2)\mu}(x) = \omega\overline{\psi}^{(1)}\Sigma^{(1,2)\mu}\psi^{(2)} + (-)^{A_1+B_1+A_2+B_2+1}\omega\overline{\psi}^{(\overline{1})}\Sigma^{(\overline{1},\overline{2})\mu}\psi^{(\overline{2})}$$
$$+ (-)^{2j+1}\left[\omega^*\overline{\psi}^{(\overline{2})}\Sigma^{(\overline{2},\overline{1})\mu}\psi^{(\overline{1})} + (-)^{A_1+B_1+A_2+B_2+1}\omega^*\overline{\psi}^{(2)}\Sigma^{(2,1)\mu}\psi^{(1)}\right] \tag{II-46}$$

Finally, to ensure proper T transformation characteristics, we form

$$j^{\mu}(x) = i^{(1,2)\mu}(x) + T\, \bar{i}^{(1,2)\mu}(-\Sigma x)\, T^{-1}$$

the effect of which, according to (II-35), is simply the addition of ω^* to ω and ω to ω^* in each of the terms in (II-46). We call $\omega + \omega^*$ the real constant Ω and finally have a hermitian Lorentz vector with the desired P and T transformations:

$$j^{\mu}(x) = \Omega\Big\{\bar{\psi}^{(1)}(x)\Sigma^{(1,2)\mu}\psi^{(2)}(x) + (-)^{A_1+B_1+A_2+B_2+1}\bar{\psi}^{(\bar{1})}(x)\Sigma^{(\bar{1},\bar{2})\mu}\psi^{(\bar{2})}(x)$$
$$+ (-)^{2j+1}\Big[\bar{\psi}^{(\bar{2})}(x)\Sigma^{(\bar{2},\bar{1})\mu}\psi^{(\bar{1})}(x) + (-)^{A_1+B_1+A_2+B_2+1}\bar{\psi}^{(2)}(x)\Sigma^{(2,1)\mu}\psi^{(1)}(x)\Big]\Big\}$$

(II-47)

Earlier the claim was made that the C-conjugation transformation need not be specifically required since we are using a normal ordered current. To show this we take matrix elements of the normal ordered current

$$\langle \vec{p}'\lambda' | :j^{\mu}(x): | \vec{p}\lambda \rangle$$

with j^{μ} given by (II-47). From (II-11), with the relation

$$\langle A_1 a_1 A_2 a_2 | A_3 a_3 \rangle = (-)^{A_1+A_2-A_3} \langle A_2 a_2 A_1 a_1 | A_3 a_3 \rangle$$

we can show

$$\Sigma^{(2,1)\mu} = (-)^{A_1+B_1+A_2+B_2+1}\, \Sigma^{(1,2)\mu} \qquad \text{(II-48)}$$

Using the fact

$$<\vec{p}'\lambda'|:\psi^{(1)}\overline{\psi}^{(2)}:|\vec{p}\lambda> = (-)^{2j} <\vec{p}'\lambda'|:\overline{\psi}^{(2)}\psi^{(1)}:|\vec{p}\lambda>$$

and (II-35), we can show

$$<\vec{p}'\lambda'|:Cj^{\mu}(x)C^{-1}:|\vec{p}\lambda> = - <\vec{p}'\lambda'|:j^{\mu}(x):|\vec{p}\lambda>$$

which is the desired C-conjugation property.

The only property required of the current left to check is the conservation condition. Here is where the field equations arise naturally. To check if $j^{\mu}(x)$ is conserved we must calculate such expressions as

$$\Sigma^{(1,2)\mu}\,\partial_{\mu}\,\psi^{(2)}(x) \quad \text{and} \quad \partial_{\mu}\,\overline{\psi}^{(1)}(x)\,\Sigma^{(1,2)\mu}$$

In calculating these expressions we make use of the relation

$$D^{1/2}(B(\vec{p}))\;D^{1/2}(B(\vec{p})) = (e^{-\psi\hat{p}\cdot\vec{\sigma}}) = \frac{p_{\mu}\sigma^{\mu}}{m}$$

$$p_{\mu}\sigma^{\mu} = m\;D^{1/2}(B(\vec{p})B(\vec{p})) \qquad \text{(II-49)}$$

as well as the Clebsch-Gordan graph techniques presented in Appendix C.[28] (For a summary of the Clebsch-Gordan coefficients and Wigner 6-j symbols, see Rotenberg et al.[29].) The results of the calculations are

$$\Sigma^{(1,2)\mu}\partial_{\mu}\psi^{(2)}(x) = -im[1/2](-)^{B_1+A_2-j-1/2}\begin{Bmatrix} A_1 & B_1 & j \\ B_2 & A_2 & 1/2 \end{Bmatrix} C^{(1)}\psi^{(1)}(x)$$

$$\text{(II-50)}$$

and from (II-48) and (II-50) follows

$$\partial_\mu \psi^{(2)}(x)\Sigma^{(2,1)\mu} = -im[1/2](-)^{B_2+A_1-j-1/2}\begin{Bmatrix} A_1 & B_1 & j \\ B_2 & A_2 & 1/2 \end{Bmatrix}\psi^{(1)}(x)C^{(1)} \tag{II-51}$$

where $\begin{Bmatrix} A_1 & B_1 & j \\ B_2 & A_2 & 1/2 \end{Bmatrix}$ is the Wigner 6-j symbol, and both (II-50) and (II-51) are also valid for antiparticle fields $\overline{\psi}$. Equations (II-50) and (II-51) enable us to show $\partial_\mu\, j^\mu(x) = 0$, with $j^\mu(x)$ given by (II-47).

What has been shown above, with the help of the free field equations (II-50) and (II-51), is that the free field current is conserved. We can use these free field equations and the minimal coupling scheme to show that the current for the interacting fields is also conserved. The 6-j symbol $\begin{Bmatrix} A_1 & B_1 & j \\ B_2 & A_2 & 1/2 \end{Bmatrix}$ is zero only when one of the four triangle inequalities involved in its definition is not satisfied, in which case $\Sigma^{(1,2)\mu}$ is also zero. For the nontrivial cases we may use (II-50) and (II-51) (along with their antiparticle counterparts) to show

$$\begin{aligned} \partial_\mu\, \Gamma^{(2,1)\mu}\, \psi^{(2,1)}(x) &= -im\, \psi^{(2,1)}(x) \\ \partial_\mu\, \tilde{\psi}^{(2,1)}(x)\, \Gamma^{(2,1)\mu} &= im\, \tilde{\psi}^{(2,1)}(x) \end{aligned} \tag{II-52}$$

where

$$\psi^{(2,1)}(x) = \begin{pmatrix} \psi^{(2)}(x) \\ \psi^{(1)}(x) \end{pmatrix} \qquad \tilde{\psi}^{(2,1)}(x) = \left(\overline{\psi}^{(2)}(x)C^{(2)}, -\overline{\psi}^{(1)}(x)C^{(1)}\right) \tag{II-53}$$

and

$$\Gamma^{(2,1)\mu} = \frac{1}{[1/2]\begin{Bmatrix} A_1 B_1 & j \\ B_2 A_2 & 1/2 \end{Bmatrix}} \begin{pmatrix} 0 & (-)^{B_2+A_1+j-1/2} C^{(2)} \Sigma^{(2,1)\mu} \\ (-)^{B_1+A_2+j-1/2} C^{(1)} \Sigma^{(1,2)\mu} & 0 \end{pmatrix}$$

(II-54)

The current (II-47) may be written as

$$j^{\mu}(x) = (-)^{B_1+A_2-j+1/2} \Omega \left\{ \tilde{\psi}^{(2,1)}(x) \; \Gamma^{(2,1)\mu} \; \psi^{(2,1)}(x) \right. \tag{II-55}$$
$$\left. + \; (-)^{2j} \; \tilde{\psi}^{(\overline{2},\overline{1})}(x) \; \Gamma^{(\overline{2},\overline{1})\mu} \; \psi^{(\overline{2},\overline{1})}(x) \right\}$$

We may use minimal coupling to obtain interacting field equations from (II-52) and show that the current (II-55) (with Ω identified with the charge of the particle through the charge form factor at zero momentum transfer) is conserved for interacting fields.

With the construction of the current (II-47) complete, the primary concern of this chapter has been met. In the remaining work we calculate the matrix elements of the current and compare with experimental data. Before we start such a calculation, however, we note from (II-47) that in general a minimum of four fields (not counting the antiparticle fields) is required for the construction of the current: (1), ($\overline{1}$), (2), and ($\overline{2}$). For spin 1/2 particles the minimum can be reduced to two fields by choosing, for example, $\overline{\psi}^{(\overline{1})}$ and $\psi^{(1)}$. However, since that is a special case of the more general current (II-47) for spin 1/2 particles, we continue to use (II-47) even after specializing to spin 1/2. We also note that the simplest possible field for a spin 1/2 particle is $\psi^{0,1/2}$ (or $\psi^{1/2,0}$), i.e., the field transforming according to the (A,B) = (0,1/2) representation. With the fields $\psi^{0,1/2}$ and $\psi^{1/2,0}$ we can construct the field equations (II-52) and the current (II-47), and find that we obtain Dirac theory. Thus, from a Lorentz group theoretic point of view, the Dirac theory is the simplest possible theory which has a chance of correctly describing the electromagnetic interactions of massive, spin 1/2 particles. With general irreducible fields, their field equations, and their currents, we are investigating the possibility that some of the more "exotic" fields may provide useful descriptions of nature. Finally, we note that by labeling the constant Ω in (II-47) as $\Omega(1,2)$, we may make the current even more general by writing

$$j^{\mu}(x) = \sum_{(1)(2)} j^{(1,2)\mu}(x)$$

We obtain (II-47) with the choice $\Omega(1,2) = \Omega$ and all other $\Omega(i,j) = 0$.

CHAPTER III

CURRENT MATRIX ELEMENTS AND FORM FACTORS

The purpose of this chapter is the actual calculation of the current matrix elements between one-particle states of the current (II-47). From such a calculation specific form factors result--that is, specific functions of the momentum transfer. To facilitate the recognition of these form factors, we first rewrite the general expression (I-45) by using the $D^{1/2}(A)$ matrices, since that is the form which obtains directly in the calculation. The calculation is made as far as is convenient for arbitrary spin, massive particles. Then the specialization to spin 1/2 is made and the form factors extracted.

Introduction

The results of the calculation of the current matrix elements are in the form of $D^{A,B}$ matrices and various functions of them. The matrix elements are labeled with a vector index, the initial and final momenta, and the initial and final polarization indices. For spin 1/2 particles this means four possible polarization combinations, and for this reason we expect the polarization indices to be associated with $D^{1/2}$ matrices and the σ^{μ} matrices. Accordingly, we rewrite the general expression (I-45) in such a manner.

First, from (I-45) we have

$$\langle p'\lambda'|j^{\mu}(0)|p\rangle = \frac{1}{(2\pi)^3} R^{\mu}(p',p)_{\lambda'\lambda}$$

where

$$R^{\mu}(p',p)_{\lambda'\lambda} = \overline{U}^{(\lambda')}(p')\left[f_1(t)\gamma^{\mu} - f_2(t)Q^{\mu}\right]U^{(\lambda)}(p) \qquad \text{(III-1)}$$

In (III-1) we have used the definitions

$$f_1(t) = F_1(t) + 2MF_2(t) \quad f_2(t) = F_2(t) \quad Q^{\mu} = p^{\mu} + p'^{\mu} \qquad \text{(III-2)}$$

The Dirac spinors are defined by (I-46) and (I-47). The usual representation of the γ matrices (defined in Appendix A) can be transformed into the antidiagonal representation

$$\Gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix} \qquad \text{(III-3)}$$

by means of

$$\Gamma^{\mu} = \xi^{\dagger}\,\gamma^{\mu}\xi \qquad \text{(III-4)}$$

where the matrix ξ is given by

$$\xi = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & \sigma^0 \\ -\sigma^0 & \sigma^0 \end{pmatrix} \qquad \text{(III-5)}$$

Use of the transformation (III-4) allows equation (III-1) to be written

$$R^{\mu}(p',p)_{\lambda'\lambda} = \overline{U}^{(\lambda')}(p')\xi\left[f_1(t)\Gamma^{\mu} - f_2(t)Q^{\mu}\right]\xi^{\dagger}\,U^{(\lambda)}(p)$$

Rewriting the Dirac spinors as

$$U^{(\lambda)}(0)_r = \begin{pmatrix} \delta_{r\lambda} \\ \\ 0 \end{pmatrix}$$

we get, with the help of (I-46),

$$U^{(\lambda)}(p)_r = \begin{pmatrix} \cosh \frac{\psi}{2}\, \delta_{r\lambda} \\ \\ \sinh \frac{\psi}{2}\, \hat{p}\cdot\vec{\sigma}_{r\lambda} \end{pmatrix}$$

where $\cosh \frac{\psi}{2} = \sqrt{(\omega+M)/2M}$ and $\sinh \frac{\psi}{2} = \sqrt{(\omega-M)/2M}$. Application of $\xi^{\dagger}$ to $U^{(\lambda)}(p)$ shows that

$$\left(\xi^{\dagger}\, U^{(\lambda)}(p)\right)_r = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{\psi}{2}\hat{p}\cdot\vec{\sigma}} \\ \\ e^{\frac{\psi}{2}\hat{p}\cdot\vec{\sigma}} \end{pmatrix}_{r\lambda} = \frac{1}{\sqrt{2}} \begin{pmatrix} D^{1/2}(B(\vec{p})) \\ \\ D^{1/2}(B(\vec{p}))^{-1} \end{pmatrix}_{r\lambda}$$

and the expression for $\overline{U}^{(\lambda')}(p')\xi$ is

$$(\overline{U}^{(\lambda')}(p')\xi)_{r'} = \frac{1}{\sqrt{2}} \left(D^{1/2}(B(\vec{p}'))^{-1}{}_{\lambda' r'}\ ,\ D^{1/2}(B(\vec{p}'))_{\lambda' r'}\right)$$

where the hermiticity of $B(\vec{p})$ has been used. Finally we have

$$R^{\mu}(p'p)_{\lambda'\lambda} = \frac{f_1(t)}{2}\left[D^{1/2}(B(\vec{p}'))\vec{\sigma}^{\mu}D^{1/2}(B(\vec{p})) + D^{1/2}(B(\vec{p}'))^{-1}\sigma^{\mu}D^{1/2}(B(\vec{p}))^{-1}\right]_{\lambda'\lambda}$$

$$- \frac{f_2(t)}{2}Q^{\mu}\left[D^{1/2}(B(\vec{p}'))^{-1}D^{1/2}(B(\vec{p})) + D^{1/2}(B(\vec{p}'))D^{1/2}(B(\vec{p}))^{-1}\right]_{\lambda'\lambda}$$

and we factor out $D^{1/2}(B(\vec{p}\,'))$ on the left and $D^{1/2}(B(\vec{p}))$ on the right to obtain

$$R^{\mu}(p',p)_{\lambda'\lambda} = D^{1/2}(B(\vec{p}\,'))_{\lambda'\alpha}\, \Gamma^{\mu}(p',p)_{\alpha\beta}\, D^{1/2}(B(\vec{p}))_{\beta\lambda}$$

$$\Gamma^{\mu}(p',p)_{\alpha\beta} = \frac{f_1(t)}{2}\left[\bar{\sigma}^{\mu} + D^{1/2}(B(\vec{p}\,')B(\vec{p}\,'))^{-1}\sigma^{\mu}\, D^{1/2}(B(\vec{p})B(\vec{p}))^{-1}\right]_{\alpha\beta}$$
$$- \frac{f_2(t)}{2}\, Q^{\mu}\left[D^{1/2}(B(\vec{p}\,')B(\vec{p}\,'))^{-1} + D^{1/2}(B(\vec{p})B(\vec{p}))^{-1}\right]_{\alpha\beta} \tag{III-6}$$

At this point we introduce two special reference frames and two Lorentz invariant kinematic variables. The two invariant variables are denoted

$$s = (p + p')^2 \qquad t = (p - p')^2 \tag{III-7}$$

and though t is the momentum transfer, s is not the usual variable denoted by s. This is because p and p' refer to an incoming and outgoing particle rather than to two incoming particles for example. (Our s is not the square of the center of mass energy.) These variables are not independent since they obey the relation

$$s + t = 4M^2 \tag{III-8}$$

The two special reference frames are the Breit frame in which $\vec{p}\,' = -\vec{p}$ and a special choice of the lab frame in which $\vec{p} = 0$ and $\vec{p}\,' = (0,0,p')$. It makes no difference in which reference frame the form factors are found since they are invariant functions, but we must be careful because the matrix elements themselves are not invariants.

Relation (II-49) may be used in (III-6) in the lab frame to show

$$\mathrm{tr}\Gamma^0(p',p)_{LAB} = \frac{s}{2M^2} G_E(t)$$

so that we have the result

$$G_E(t) = \frac{2M^2}{s} \mathrm{tr}\Gamma^0(p',p)_{LAB} \qquad \text{(III-9)}$$

The same function for $G_E(t)$ must result regardless of the reference frame used, but in frames other than the lab frame the expressions obtained are not necessarily of the same form as (III-9). For example, in the Breit frame with $\vec{p} = (0,0,p)$ the expression is

$$G_E(t) = \frac{1}{2} \mathrm{tr}\Gamma^0(p',p)_{B.F.}$$

There are several ways of finding expressions for the magnetic form factor $G_M(t) = f_1(t)$, but only one proves to be simple when finding $G_M(t)$ from the calculated matrix elements. In the lab frame, expression (III-6) enables us to show

$$\mathrm{tr}\left(\sum_\mu \Gamma^\mu(p',p)_{LAB}\bar{\sigma}^\mu\right) = 2G_E(t) + \frac{t}{M^2} G_M(t) \qquad \text{(III-10)}$$

Matrix Elements

We are now ready to calculate the matrix elements

$$\langle p'\lambda'|j^{\mu}(0)|p\lambda\rangle = \frac{1}{(2\pi)^3} R^{\mu}(p',p)_{\lambda'\lambda}$$

with the current j^{μ} given by (II-47). Since the four terms in j^{μ} are identical in form, with only phases and representation labels being different, only one of the terms is specifically calculated. The remaining terms are obtained by inspection. We choose to work with the first term and evaluate the matrix element

$$\langle p'\lambda'|j^{\mu(1,2)}(0)|p\lambda\rangle = \Omega \langle p'\lambda'|\overline{\psi}^{(1)}(x)\ \Sigma^{(1,2)\mu}\ \psi^{(2)}(x)|p\lambda\rangle \qquad \text{(III-11)}$$

We use (A_1,B_1) and (A_2,B_2) for the representations (1) and (2). With these labels (III-11) becomes

$$\frac{1}{(2\pi)^3} R^{\mu}{}^{(A_1B_1,A_2B_2)}_{(p',p)_{\lambda'\lambda}} = \Omega \langle 0|a(\vec{p}',\lambda')\overline{\psi}^{A_1B_1}(0)_{a_1b_1}\psi^{A_2B_2}(0)_{a_2b_2}a^{\dagger}(p,\lambda)|0\rangle$$

$$\times \Sigma^{A_1B_1,A_2B_2:\mu}_{a_1b_1,a_2b_2} \qquad \text{(III-12)}$$

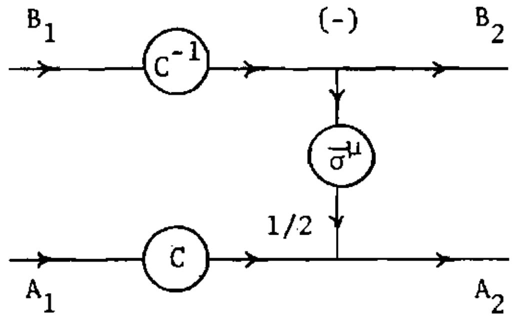
Because of normal ordering only the positive frequency part of $\psi^{A_2B_2}$ and the negative frequency part of $\overline{\psi}^{A_1B_1}$ contribute to the matrix element (all other combinations giving zero). With the fields given by (II-31) and (II-32), and the commutation relations (II-33), the result of (III-12) is

$$R^{(A_1B_1,A_2B_2)}_{(p',p)_{\lambda'\lambda}} = \Omega(-)^{2A_1} C_{\lambda'\alpha} \langle j\alpha | A_1 a_1' B_1 b_1' \rangle D^{A_1B_1}(B(\vec{p}\,'))^T_{a_1'b_1',a_1b_1}$$

$$\times \sum\nolimits^{A_1B_1,A_2B_2:\mu}_{a_1b_1,a_2b_2} D^{A_2B_2}(B(\vec{p}))_{a_2b_2,a_2'b_2'} \langle A_2a_2'B_2b_2' | j\lambda \rangle \qquad \text{(III-13)}$$

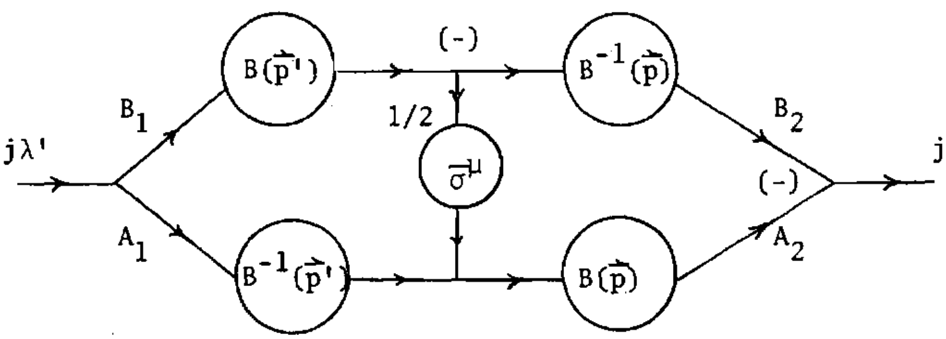
At this point it is convenient to write $\sum\nolimits^{A_1B_1,A_2B_2:\mu}_{a_1b_1,a_2b_2}$ in graphical form using the graph techniques of Appendix C, with the result

$$\sum\nolimits^{A_1B_1,A_2B_2:\mu}_{a_1b_1,a_2b_2} = [1/2](-)^{2A_1}$$



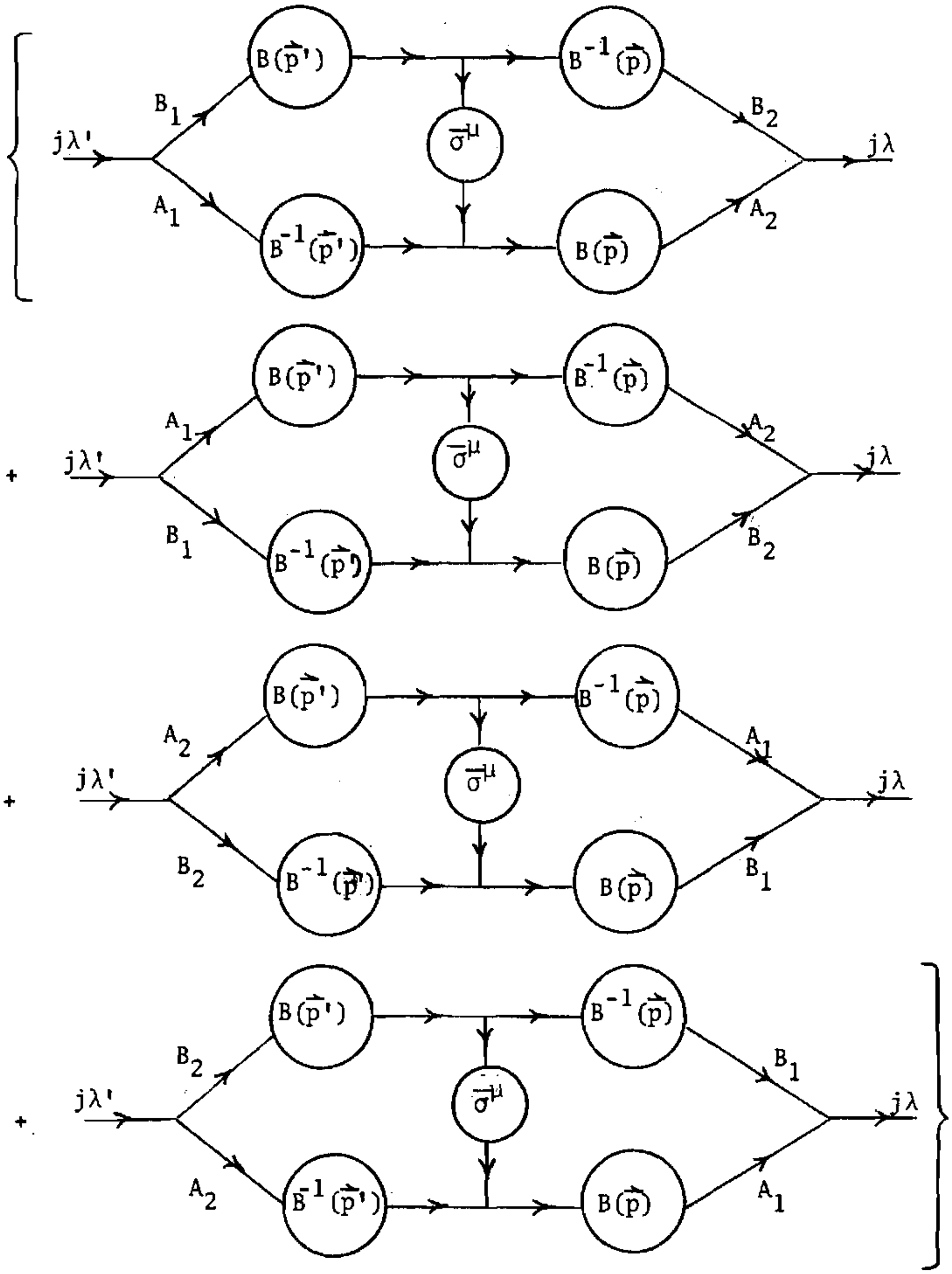
Now the graphical representation of (III-13) can be easily shown to be

$$R^{\mu\,(A_1B_1,A_2B_2)}_{(p',p)_{\lambda'\lambda}} = \Omega(-)^{2A_1} [j] [1/2] \qquad \text{(III-14)}$$



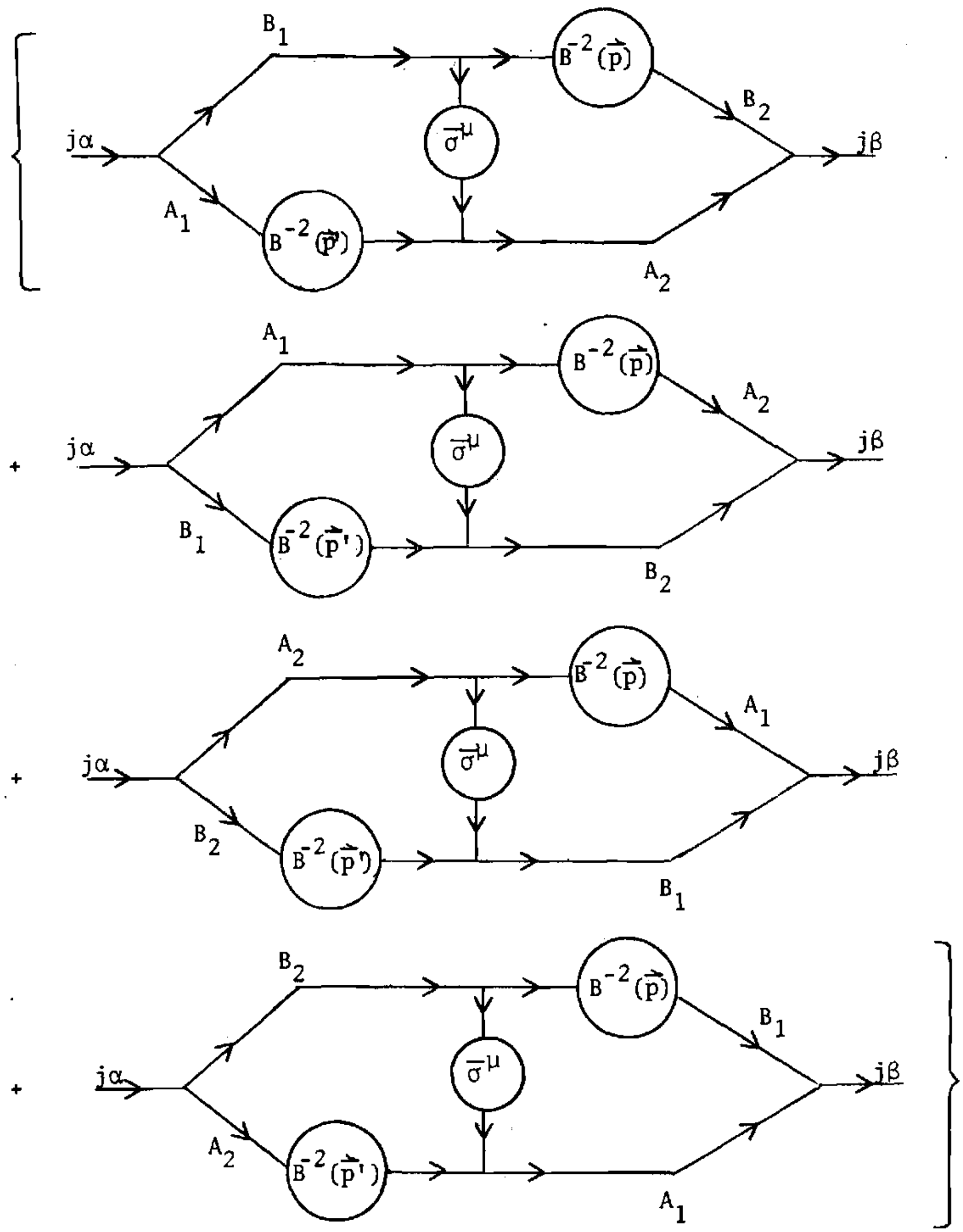
The hermiticity of $B(\vec{p}\,')$ has been used in (III-14). The other three terms in the current matrix element can be written down by inspection of (II-47) and (III-14); and when the (-) indicators have all been removed from the vertices and all phase factors collected, there results

$$R^{\mu}(p',p)_{\lambda'\lambda} = \Omega(-)^{B_1-A_2+\frac{1}{2}-j}\,[j]\,[\tfrac{1}{2}] \tag{III-15}$$



We note that (III-15) is correct for arbitrary spin j. Finally, we have

$$\Gamma^{\mu}(p',p)_{\alpha\beta} = \Omega(-)^{B_1-A_2+\frac{1}{2}-j}\,[j]\,[\tfrac{1}{2}] \qquad \text{(III-16)}$$

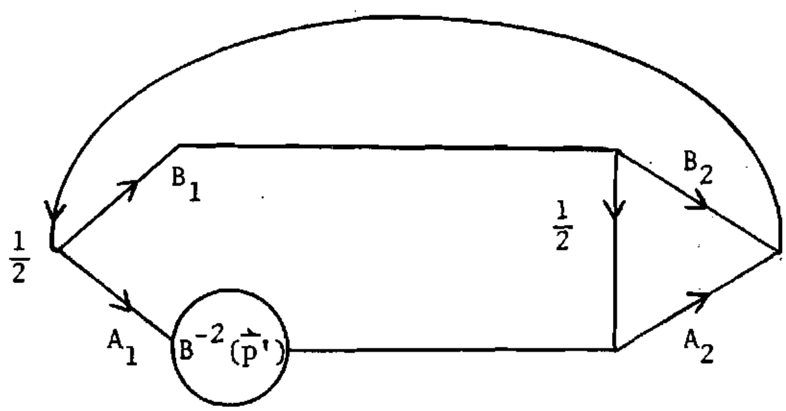


where the notation $B(\vec{p})B(\vec{p}) = B^2(\vec{p})$ is used.

With the result (III-16), the calculation of the current matrix element is complete. The form factors for spin 1/2 particles may now be extracted with the use of (III-9) and (III-10). Before extracting the form factors, however, we point out that with patience it can be shown that expression (III-16) does in fact have the form required by (III-6). The proof is long and involved, most easily accomplished in the Breit frame (with $-\vec{p}' = \vec{p} = (0,0,p)$), and does not add to the work except as a check on the result (III-16).

Form Factors

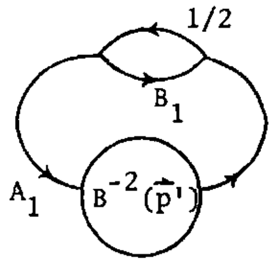
We now specialize (III-16) to the case of spin 1/2 particles and extract the form factors. The easiest to find is $G_E(t)$, and the most convenient frame for the calculation is the lab frame with $\vec{p} = 0$ and $\vec{p}' = (0,0,p')$. In this case $D(B(\vec{p})) = 1$, and $D(B(\vec{p}'))$ is diagonal. Since all four terms in (III-16) are identical in form, only one is calculated, the others later are written down by inspection. Thus we consider the trace of the first graph with $\mu = 0$:



(III-17)

The triangle in (III-17) may be collapsed according to the rules in Appendix C resulting in

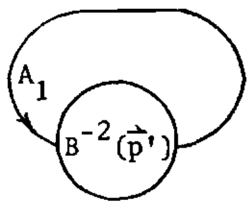
$$(-)^{A_1+B_1+\frac{1}{2}} \begin{Bmatrix} A_1 & B_1 & \frac{1}{2} \\ B_2 & A_2 & \frac{1}{2} \end{Bmatrix}$$



(III-18)

The "bubble" in (III-18) may be collapsed (Appendix C) yielding

$$\frac{(-)^1}{[A_1]} \begin{Bmatrix} A_1 & B_1 & \frac{1}{2} \\ B_2 & A_2 & \frac{1}{2} \end{Bmatrix}$$



$$= \frac{(-)^1}{[A_1]} \begin{Bmatrix} A_1 & B_1 & \frac{1}{2} \\ B_2 & A_2 & \frac{1}{2} \end{Bmatrix} \text{ tr } D^{A_1}(B^{-2}(\vec{p}'))$$

(III-19)

The remaining terms are identical to (III-19) in form, with the result ($[j] = [1/2]$)

$$\text{tr } \Gamma^0(p',p)_{LAB} = (-)^{B_1-A_2+1} [\tfrac{1}{2}]^2 \begin{Bmatrix} A_1 & B_1 & \frac{1}{2} \\ B_2 & A_2 & \frac{1}{2} \end{Bmatrix}$$

(III-20)

$$\left\{ \frac{\text{tr } D^{A_1}(B^{-2}(\vec{p}'))}{[A_1]} + \frac{\text{tr } D^{B_1}(B^{-2}(\vec{p}'))}{[B_1]} + \frac{\text{tr } D^{A_2}(B^{-2}(\vec{p}'))}{[A_2]} + \frac{\text{tr } D^{B_2}(B^{-2}(\vec{p}'))}{[B_2]} \right\}$$

Substitution of (III-20) into (III-9) yields for the electric form factor

$$G_E(t) = \Omega(-)^{B_1-A_2+1} \begin{Bmatrix} A_1 & B_1 & \frac{1}{2} \\ B_2 & A_2 & \frac{1}{2} \end{Bmatrix} \frac{8M^2}{s}$$

(III-21)

$$\left\{ \frac{T^{(A_1)}(t)}{[A_1]} + \frac{T^{(B_1)}(t)}{[B_1]} + \frac{T^{(A_2)}(t)}{[A_2]} + \frac{T^{(B_2)}(t)}{[B_2]} \right\}$$

In (III-21) the notation

$$T^{(j)}(t) = \mathrm{tr}\ D^j(B^{-2}(\vec{p}\,')) \tag{III-22}$$

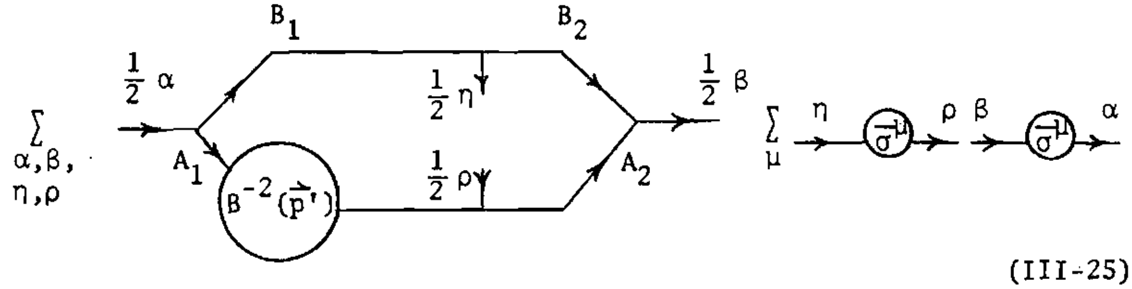
has been introduced, it being understood that $T^{(j)}(t)$ is an invariant function, but also that the right-hand side of (II-22) must be evaluated in the lab frame to give the correct function for $T^{(j)}(t)$. The invariant notation in (III-21) is valid since $s = 4M^2 - t$ and since $B^{-2}(\vec{p}\,'_{LAB})$ can be written in the invariant form (remember that $B(\vec{p}\,')$ is an element of SL(2,c))

$$B^{-2}(\vec{p}\,'_{LAB}) = \begin{pmatrix} \dfrac{2M^2-t+\sqrt{t^2-4M^2t}}{2M^2} & 0 \\ 0 & \dfrac{2M^2-t-\sqrt{t^2-4M^2t}}{2M^2} \end{pmatrix} \tag{III-23}$$

The extraction of the magnetic form factor is more involved than that of the electric form factor. First, we need to calculate

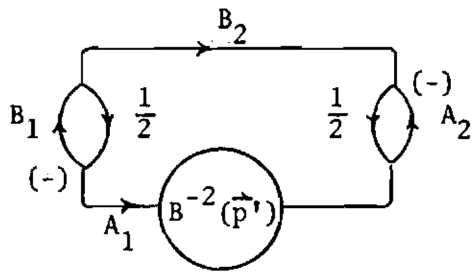
$$\mathrm{tr}(\sum_{\mu} \Gamma^{\mu}(p',p)_{LAB}\ \overline{\sigma}^{\mu}) \tag{III-24}$$

As in the previous calculation we only work with one of the terms in $\Gamma^{\mu}(p',p)$. We consider the first term and calculate



(III-25)

The rules in Appendices B and C allow (III-25) to be simplified to the expression

2 

(III-26)

The bubbles in (III-26) may be collapsed resulting in the expression

$$2\delta_{A_1B_2}(-)^{2B_2}\,\frac{\mathrm{tr}\,D^{A_1}(B^{-2}(\vec{p}\,'))}{[A_1]}$$

$$\times\ \Delta(A_1\ B_1\ \tfrac{1}{2})\ \Delta(A_2\ B_2\ \tfrac{1}{2})\ \Delta(A_1\ A_2\ \tfrac{1}{2})\ \Delta(B_1\ B_2\ \tfrac{1}{2}) \qquad \text{(III-27)}$$

The Δ's are included in (III-27) because the Clebsch-Gordan coefficients actually contain them. We do not continue to carry these Δ's, but they should be understood to be present. When we collect all four terms the result is

$$\mathrm{tr}(\sum_{\mu}\Gamma^{\mu}(p',p)_{LAB}\overline{\sigma}^{\mu}) = 8\Omega(-)^{B_1-A_2}\left\{\frac{\delta_{A_1B_2}(-)^{2A_1}}{[A_1]}\,\frac{T^{(A_1)}(t)}{[A_1]}\right.$$

$$\left.+\frac{\delta_{B_1A_2}(-)^{2B_1}}{[B_1]}\,\frac{T^{(B_1)}(t)}{[B_1]}+\frac{\delta_{A_2B_1}(-)^{2A_2}}{[A_2]}\,\frac{T^{(A_2)}(t)}{[A_2]}+\frac{\delta_{B_2A_1}(-)^{2B_2}}{[B_2]}\,\frac{T^{(B_2)}(t)}{[B_2]}\right\}$$

(III-28)

Here it is convenient to point out the relation (see Rotenberg et al.[29])

$$\begin{Bmatrix} A_1 & B_1 & \frac{1}{2} \\ B_2 & A_2 & \frac{1}{2} \end{Bmatrix} = \left[\frac{(-)^{2A_1+1}}{[A_1]} \delta_{A_1 B_2} + \frac{(-)^{2B_1+1}}{[B_1]} \delta_{B_1 A_2} \right]$$

$$\times\, \Delta(A_1\ B_1\ \tfrac{1}{2})\ \Delta(A_2\ B_2\ \tfrac{1}{2})\ \Delta(A_1\ A_2\ \tfrac{1}{2})\ \Delta(B_1\ B_2\ \tfrac{1}{2}) \tag{III-29}$$

With the help of (III-10), (III-21), and (III-29), we can show that the magnetic form factor is given by

$$G_M(t) = \Omega(-)^{B_1+A_2+1} \frac{16M^2}{s} \left\{ - \frac{\delta_{A_1 B_2}}{[A_1]} \frac{T^{(A_1)}(t)}{[A_1]} + \frac{\delta_{B_1 A_2}}{[B_1]} \frac{T^{(B_1)}(t)}{[B_1]} \right.$$

$$\left. + \left(\frac{\delta_{B_1 A_2}}{[B_1]} + \frac{\delta_{A_1 B_2}}{[A_1]} \right) \frac{M^2}{t} \left[\frac{T^{(A_1)}(t)}{[A_1]} - \frac{T^{(B_1)}(t)}{[B_1]} - \frac{T^{(A_2)}(t)}{[A_2]} + \frac{T^{(B_2)}(t)}{[B_2]} \right] \right\} \tag{III-30}$$

The expressions (III-21) and (III-30) for the form factors are somewhat complex, but the evaluation of these functions is simplified by the repeated appearance of the functions $T^{(j)}(t)/[j]$. These functions are themselves rather complex, but we need only consider the two cases of integral spin and half-integral spin for one of these functions. However, even before these functions are explicitly evaluated we can see from (III-23) and (B-5) that they are essentially polynomials in t.

In the first Weinberg paper[26] it is shown that the matrix

$$\pi^{(j)}(\vec{p}) = m^{2j}\; D^j(B(\vec{p}))\; D^j(B(\vec{p}))$$

is a polynomial in the momentum vector p^μ. Expressions are given for integral j and for half-integral j, and the traces of these matrices can be taken. For convenience we introduce the variable $x = -t/(4M^2)$, and in terms of the variable x the traces result in

$$\frac{T^{(j)}(x)}{[j]} = 1 + \frac{8}{3}\, j(j+1)x(1+x) + \sum_{n=1}^{j-1} \frac{1}{2n+3} \begin{pmatrix} j+n+1 \\ 2n+2 \end{pmatrix} 4^{2n+2}\, x^{n+1}\, (1+x)^{n+1} \tag{III-31}$$

$$\frac{T^{(j+\frac{1}{2})}}{[j+\frac{1}{2}]} = 1 + 2x + \frac{2}{[j+\frac{1}{2}]} \sum_{n=1}^{j} \begin{pmatrix} j+n+1 \\ 2n+1 \end{pmatrix} 4^{2n}(1+2x)\; x^n(1+x)^n \tag{III-32}$$

In (III-31) and (III-32) j is an integer; and when the upper limit on the summation is less than the lower limit, the term involving the summation is zero. The $\begin{pmatrix} n \\ m \end{pmatrix}$ are binomial coefficients. Table 2 shows the function $T^{(j)}(x)/[j]$ for $j \leqq 3$.

It is evident from expressions (III-21), (III-30), (III-31), and (III-32) that the form factors are polynomials in $x = -t/(4M^2)$. It is not necessary to rewrite (III-31) and (III-32) in the form of a simple power series in x. Rather, we shall investigate the form factors and their derivatives at x = 0 and can compare with experimental data.

Before listing the results for the form factors we introduce

Table 2. The Function $\frac{T^{(j)}(x)}{[j]}$ for $j \leqq 3$

j	$\frac{T^{(j)}(x)}{[j]}$
0	1
$\frac{1}{2}$	$2x + 1$
1	$\frac{16}{3}x^2 + \frac{16}{3}x + 1$
$\frac{3}{2}$	$\frac{64}{4}x^3 + \frac{96}{4}x^2 + \frac{40}{4}x + 1$
2	$\frac{256}{5}x^4 + \frac{512}{5}x^3 + \frac{336}{5}x^2 + \frac{80}{5}x + 1$
$\frac{5}{2}$	$\frac{1024}{6}x^5 + \frac{2560}{6}x^4 + \frac{2304}{6}x^3 + \frac{896}{6}x^2 + \frac{140}{6}x + 1$
3	$\frac{4096}{7}x^6 + \frac{12,288}{7}x^5 + \frac{14,080}{7}x^4 + \frac{7680}{7}x^3 + \frac{2016}{7}x^2 + \frac{224}{7}x + 1$

some compact notation. We label the form factors in (III-21) and (III-30) as

$$G_E^{(A_1,B_1;A_2,B_2)}(t) \qquad G_M^{(A_1,B_1;A_2,B_2)}(t) \tag{III-33}$$

We recall that expression (III-30) is actually multiplied by $\Delta(A_1B_1 1/2)\ \Delta(A_2B_2 1/2)\ \Delta(A_1A_2 1/2)\ \Delta(B_1B_2 1/2)$, and the definition of the 6-j symbol $\begin{Bmatrix} A_1 B_1 1/2 \\ B_2 A_2 1/2 \end{Bmatrix}$ also contains these same four Δ's. If we choose $B_1 = B$ we can list all possible coupling combinations as follows:

$$(A_1,B_1;A_2,B_2) = \begin{cases} (B-\frac{1}{2},B;B,B-\frac{1}{2}) \\ (B-\frac{1}{2},B;B,B+\frac{1}{2}) \\ (B-\frac{1}{2},B;B-1,B-\frac{1}{2}) \\ (B+\frac{1}{2},B;B,B-\frac{1}{2}) \\ (B+\frac{1}{2},B;B,B+\frac{1}{2}) \\ (B+\frac{1}{2},B;B+1,B+\frac{1}{2}) \end{cases} \tag{III-34}$$

Because of the symmetries involved in expressions (III-21) and (III-30) (in fact the symmetries built into the current), these six possible combinations can all be included in the two combinations

$$\begin{aligned} (B^-) &= (B-\tfrac{1}{2},B;B,B-\tfrac{1}{2}) \\ (B^+) &= (B-\tfrac{1}{2},B;B,B+\tfrac{1}{2}) \end{aligned} \tag{III-35}$$

Use of the symmetries to rearrange (III-35) in general introduces phase factors into the expressions for G_E and G_M. However, the phase factors introduced are the same for G_E as for G_M, thus producing an overall phase factor which may as well be absorbed into Ω.

There are four cases to be considered for each form factor: $G^{(B^\pm)}$ with B integral or half-integral. We list here the results of just two cases. All eight expressions (four for G_E and four for G_M) are somewhat similar, and nothing is to be gained by listing them all. The following are the expressions for the electric form factor for the half-integral (B^-) case $G_E^{(B^-)}(x)^{H.I.}$ and the magnetic form factor for the integral (B^+) case $G_M^{(B^+)}(x)^{I.}$:

$$G_E^{(B^-)}(x)^{H.I.} = \frac{2(-)^{2B+1}}{B(2B+1)}\,\Omega^{(B^-)}\left\{2 + \frac{8}{3}(B-\tfrac{1}{2})\,(B+\tfrac{1}{2})x + \sum_{n=1}^{B-\frac{3}{2}} \frac{1}{2n+3}\begin{pmatrix} B+n+\frac{1}{2} \\ 2n+2 \end{pmatrix} 4^{2n+2}\, x^{n+1}(1+x)^n + \frac{2}{2B+1}\sum_{n=1}^{B-\frac{1}{2}} \begin{pmatrix} B+n+\frac{1}{2} \\ 2n+1 \end{pmatrix} 4^{2n}(1+2x)x^n(1+x)^{n-1}\right\} \qquad \text{(III-36)}$$

$$G_M^{(B^+)}(x)^{I.} = \frac{4(-)^{2B+1}}{2B+1}\,\Omega^{(B^+)}\left\{\frac{4}{3}B(B+1) + \frac{8}{3}B(B+1)x + \frac{1}{2}\sum_{n=1}^{B-1}\frac{1}{2n+3}\begin{pmatrix} B+n+1 \\ 2n+2 \end{pmatrix} 4^{2n+2}(1+2x)x^n(1+x)^n - \frac{1}{4B}\sum_{n=1}^{B-1}\begin{pmatrix} B+n \\ 2n+1 \end{pmatrix} 4^{2n}(1+2x)x^{n-1}(1+x)^{n-1} - \frac{1}{4(B+1)}\sum_{n=1}^{B}\begin{pmatrix} B+n+1 \\ 2n+1 \end{pmatrix} 4^{2n}(1+2x)x^{n-1}(1+x)^{n-1}\right\} \qquad \text{(III-37)}$$

It is apparent from (III-36) that $G_E^{(B^-)}(x)^{H.I.}$ is a monotonic function of x. In fact it is easily seen from (III-21), (III-31), and (III-32) that all four cases for G_E are monotonic functions. It is not so easy to observe the character of $G_M^{(B^+)}(x)^{I.}$ from (III-37). However, for our purposes it is enough to know the character of $G_E(x)$.

Table 3 shows a list of the form factors up to the (2^+) coupling. In each coupling case the coupling constant Ω is chosen so that $G_E(0) = 1$ (the coupling constant is the same for G_E and G_M). Table 4 shows the t = 0 values of the form factors and their derivatives. Here again the constant Ω is chosen for each case so that $G_E(0) = 1$, but here the results are valid for general values of B in the coupling $(B^{\pm})$. Figure 7 shows some of the form factors up to the (2^-) coupling plotted for $0 \leq x \leq 1$. The curves for $(1/2^+)$, (1^+), and $(3/2^+)$ are multiplied by -1.

With the expressions for $G_E(x)$ and $G_M(x)$ and with Tables 3 and 4 completed, the work of this chapter is finished. We leave the discussion of the results to the final chapter.

Table 3. Form Factors for Couplings up to (2^+). $x = \frac{-t}{4M^2}$

$(A_1, B_1; A_2 B_2) = (B^{\pm})$	$G_E(x)$	$G_M(x)$
$(0\ \frac{1}{2}\ \frac{1}{2}\ 0) = (\frac{1}{2}^-)$	1	1
$(0\ \frac{1}{2}\ \frac{1}{2}\ 1) = (\frac{1}{2}^+)$	$\frac{1}{3}(4x + 3)$	$-\frac{1}{3}$
$(\frac{1}{2}\ 1\ 1\ \frac{1}{2}) = (1^-)$	$\frac{1}{3}(8x + 3)$	$\frac{1}{3}(16x + 11)$
$(\frac{1}{2}\ 1\ 1\ \frac{3}{2}) = (1^+)$	$\frac{1}{3}(12x^2 + 14x + 3)$	$-\frac{1}{3}(2x + 1)$
$(1\ \frac{3}{2}\ \frac{3}{2}\ 1) = (\frac{3}{2}^-)$	$\frac{1}{3}(24x^2 + 20x + 3)$	$\frac{1}{3}(72x^2 + 88x + 23)$
$(1\ \frac{3}{2}\ \frac{3}{2}\ 2) = (\frac{3}{2}^+)$	$\frac{1}{15}(192x^3 + 312x^2 + 140x + 15)$	$-\frac{1}{15}(24x^2 + 24x + 5)$
$(\frac{3}{2}\ 2\ 2\ \frac{3}{2}) = (2^-)$	$\frac{1}{5}(128x^3 + 168x^2 + 60x + 5)$	$\frac{1}{5}(512x^3 + 888x^2 + 456x + 65)$
$(\frac{3}{2}\ 2\ 2\ \frac{5}{2}) = (2^+)$	$\frac{1}{15}(640x^4 + 1344x^3 + 924x^2 + 230x + 15)$	$-\frac{1}{15}(64x^3 + 96x^2 + 42x + 5)$

CHAPTER IV

DISCUSSION

In this chapter a discussion of the results of the investigation is presented. Comparisons are made between the calculated form factors and experimental data for electrons, muons, protons, and neutrons. Other areas of interest which arose as a result of this investigation or preparation for it are also briefly mentioned. However, before the discussion of any specific applications, the results are considered in a general manner.

When the Dirac field is used for the description of massive, spin 1/2 particles (distinct from their antiparticles) and is minimally coupled to the electromagnetic field, the dynamics uniquely leads to constant form factors $G_E(x) = G_M(x) = 1$. Such constant form factors are indicative of point particles (or structureless particles) concerning the electromagnetic interaction. Furthermore, with the Dirac field minimally coupled to the electromagnetic field there is no possibility of describing nontrivial electromagnetic interactions of any of the observed chargeless particles. If we use more exotic fields (which are allowed by relativistic kinematics and symmetries) than the Dirac field, however, minimal coupling does lead to electromagnetic structure, as evidenced by the form factors in Table 3. The point of interest here is that electromagnetic structure can be generated from the dynamics of minimal coupling if fields other than the simplest possible are used.

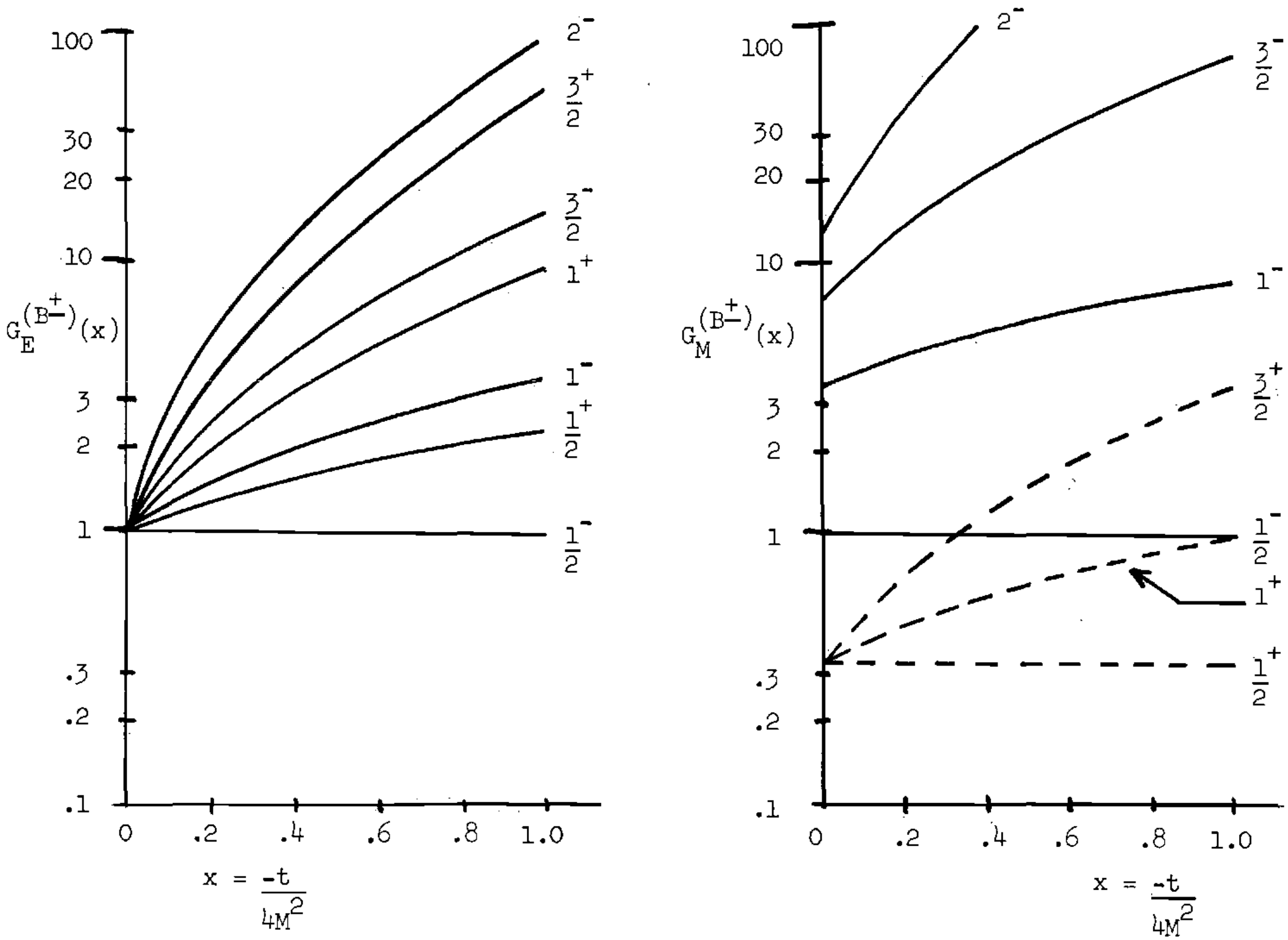


Figure 7. Form Factors

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The relation between the charge of a particle and its electric form factor G_E is

$$q = \pm e\, G_E(0) \tag{IV-1}$$

where the $\pm$ determines the sign of the charge q. From (IV-1) we see the possibility of a chargeless particle interacting electromagnetically even if minimal coupling is used. The requirement here is that $G_E(x)$ satisfies the condition

$$G_E(0) = 0$$

Thus, in order to describe an electromagnetically interacting chargeless particle using minimal coupling, we must have an electric form factor which vanishes at x = 0, and in addition one or both of G_E and G_M must not vanish identically.

The coupling scheme represented by $(\frac{1}{2}^-)$ is the same as the Dirac theory, and from Table 3 it is seen that the correct form factors result in this case. Such a coupling evidently correctly describes both the electron and muon. The anomalous magnetic moments of the charged leptons to first order in the interaction are determined by the a factor

$$a = \frac{g - 2}{2} = 2MF_2(0) = G_M(0) - G_E(0) = 0$$

The experimentally determined values of a for the leptons are non-zero, but they are satisfactorily explained by the renormalization programs in interactions of order higher than the first.

The results are not so encouraging when the nucleons are considered. From Figure 3 we see that the proton electric form factor $G_{EP}(x)$ has the value 1 at $x = 0$ and decreases for increasing x in a nonpolynomial manner (the dipole fit is the reciprocal of a polynomial of finite order). The asymptotic behavior of $G_{EP}(x)$ is not clear from Figure 3 but certainly appears as if it will not cross zero. The same description holds for the proton magnetic form factor $G_{MP}(x)$, with the exception that at $x = 0$ $G_{MP}(x)$ has the value $\mu_P \cong 2.79$. Figure 4 shows the data for the neutron; and the magnetic form factor behaves in a manner similar to both proton form factors, except its $x = 0$ value is $\mu_N \cong -1.91$. The data for $G_{En}(x)$ show a value of zero for $x = 0$, and are consistent with a small increase in $G_{En}(x)$ for small x.

In attempting to fit the nucleon data with our calculated form factors it is immediately obvious that coupling schemes of the type $(B^{\pm})$ will not work. This is evident from the facts that (1) $G_E^{(B^{\pm})}(x)$ is a monotonic polynomial in $x = \frac{-t}{4M^2}$ and (2) $G_E^{(B^{\pm})}(0)$ is not zero for any B. The nonvanishing of $G_E^{(B^{\pm})}(0)$ means that the simplest minimal coupling schemes $(B^{\pm})$ will not describe the neutron, nor for that matter any other chargeless particle. The monotonic polynomial nature of G_E means also that the proton cannot be described so simply since none of the $G_E^{(B^{\pm})}$ start from 1 at $x = 0$ and decrease with increasing x. This behavior of $G_E^{(B^{\pm})}$ is the reason we have not attempted to establish the nature of the polynomials for $G_M^{(B^{\pm})}$. Thus, it is not possible to satisfactorily describe even the low momentum transfer behavior of protons and neutrons without at least using combinations of $G^{(B^{+}_{-})}$ for different B values.

The combination of different currents was mentioned at the end of Chapter II. Here we introduce the notation $j^{(B^{+}_{-})}(x)$ for the current obtained from the (B^{+}_{-}) coupling with the coupling constant chosen so that $G_E^{(B^{+}_{-})}(0) = 1$. We can write a more general expression for the current as

$$J^{\mu}(x) = \sum_{B=\frac{1}{2},1,\ldots} \Omega_{(B^{+}_{-})}\, j^{(B^{+}_{-})\mu}(x) \tag{IV-2}$$

The $\Omega_{(B^{+}_{-})}$ are adjustable constants subject to the condition

$$\sum_B \Omega_{(B^{+}_{-})} = q$$

where q is the charge of the particle being described. The current given by (IV-2) leads to form factors given by

$$G_{E,M}(x) = \sum_B \Omega_{(B^{+}_{-})}\, G_{E,M}^{(B^{+}_{-})}(x) \tag{IV-3}$$

Since the functions $G_E^{(B^{+}_{-})}(x)$ and $G_M^{(B^{+}_{-})}(x)$ are polynomials in x ($G_E^{(B^{+}_{-})}(x)$ is of order $2B^{+}$ or $2B^{-} - 1$), any well-behaved function, such as the dipole fit to the nucleon form factors, can be fit by the expression (IV-3). However, the dipole fit would require (IV-3) to be an infinite series, and we certainly do not feel that such a description of the nucleon (a direct sum of an infinite number of irreducible representations with an infinite number of parameters adjusted suitably) would be any simplification. Additionally, since the $\Omega_{(B^{+}_{-})}$ must be the same for G_E as for G_M, it is certainly not clear that the fitting of $G_E(x)$ would also produce a fit to $G_M(x)$.

Certainly we should not expect to fit the nucleon form factors for all momentum transfers since for large values of -t the vector meson resonances surely play an important role (Figure 6). However, for low momentum transfer such resonances should not contribute appreciably, and we might anticipate the possibility of explaining nucleon structure via the $(B^{\pm})$ couplings. The combination of two currents must be one of the three forms

$$(B^-) + (A^-) \qquad (B^-) + (A^+) \qquad (B^+) + (A^+) \tag{IV-4}$$

For one particle there are four conditions to be met at x = 0: G_E, G_E', G_M, and G_M'. In the combination of two coupling schemes there occur four parameters: $\Omega_{(B)}$, $\Omega_{(A)}$, B, and A. From Table 4, however, it can be shown that neither the proton nor the neutron can be fit by any of the possibilities in (IV-4). For either particle, one or the other of G_E and G_M may easily be fit in any number of ways for small x, but G_E and G_M cannot both be fit. If we use more than two coupling schemes in combination, we meet the unsatisfactory situation of having more adjustable parameters than conditions to be met. Fitting data with excessive numbers of adjustable parameters does not come under the heading of an explanation.

From the preceding discussion we conclude that neither of the nucleons can be satisfactorily explained with regard to low momentum transfer electromagnetic structure using finite component fields minimally coupled to the electromagnetic field and lowest order perturbation theory. The best that can result from such a treatment seems to be the possibility of fitting the nucleon form factors simply by using enough adjustable parameters.

Of the restrictions mentioned in the previous paragraph (low momentum transfer, finite component fields, minimal coupling, and perturbation theory), perhaps the one we should first suspect of leading to the essentially negative results is the perturbation treatment. We have used first-order, nondegenerate perturbation theory in this calculation. We expected that for low energy electron-nucleon scattering (necessarily requiring low momentum transfer) perturbation theory should be applicable. The difficulty here lies in the use of the (A,B) representations of SL(2,c) with neither A nor B having the value zero. Under rotations the representation $D^{A,B}(R)$ is reducible into representations with spins j satisfying the triangle inequality $|A-B| \leqq j \leqq A+B$. We have used the Clebsch-Gordan coefficients to project out the specific spin ($j = 1/2$) of interest and therefore have neglected the other possible spins associated with the (A,B) representation. A wave function containing a part which transforms according to the (A,B) representation can be made to satisfy a linear wave equation with a mass spectrum resulting for the various spin states contained.[30] It is found[30] that the mass spectrum may be richly populated near the low mass end. Thus, there may be other states contained in the (A,B) representation which are quite close in energy to the lowest state. In the course of interactions, then, these states might easily become mixed; and in such a case we should have to use degenerate perturbation theory.

During the course of this investigation and preparation for it, several related problems arose. Since expression (III-15) for the matrix elements of the current is valid for arbitrary j, it would be of interest to extend the calculation to values of j different from 1/2. Of course for spins greater than 1/2 the calculation will not be so easy as for the spin 1/2 case. The zero spin case, however, should be simple and might be compared with the pions for example. The results of the calculations in this work would certainly lead us to expect polynomials in the other cases also, however. Another further investigation, one which at first sight appears very interesting, is the consideration of short-range electromagnetic interactions (see, for example, Weinberg[31]). The possibility of short-range electromagnetic interactions arises in the Lorentz group theoretic treatment of the spin 1, massless photon. In order to obtain an irreducible field (one transforming with no gauge functions introduced) for the photon, it is found that the photon must be described by nonvector fields (the (1/2,1/2) representation), the simplest of which ((0,1) and (1,0)) lead to interactions which could not produce long-range effects. The possibility that such short-range interactions, perhaps in combination with the ordinary vector potential interaction, might lead to different electromagnetic structure than resulted in this work seems great, and this is one more avenue in the search for explanations of electromagnetic interactions of elementary particles.

APPENDIX A

NOTATION

The Heaviside-Lorentz system of electromagnetic units is used throughout this work, along with natural units in which $\hbar$ and c are equal to one. The symbol e, when used as a charge, means an elementary positive charge, i.e., the charge of the electron is -e.

The metric defined by

$$g_{00} = 1 \qquad g_{ij} = -\delta_{ij}$$

$$g_{0i} = g_{i0} = 0$$

is used, as well as the summation convention of summing on repeated indices, except as otherwise indicated. The space-time position vector is defined by

$$x^{\mu} = (x^0, x^1, x^2, x^3) = (t, x, y, z) = (t, \vec{x})$$

and

$$x_{\mu} = g_{\mu\lambda}\, x^{\lambda} = (t, -\vec{x})$$

The energy-momentum vector is defined similarly:

$$p^{\mu} = (E, \vec{p}) = (\omega, \vec{p})$$

and the momentum operator in the space-time representation is

$$p^{\mu} = i\partial^{\mu} = \left(\frac{i\partial}{\partial t}, \frac{1}{i}\vec{\nabla}\right)$$

where the notation

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad \text{and } \partial^{\mu} = g^{\mu\lambda}\, \partial_{\lambda}$$

is used. The scalar product of two vectors a and b is defined as

$$a \cdot b = a^{\mu}\, b_{\mu} = a^{\mu}\, b^{\lambda}\, g_{\mu\lambda}$$

Integrals over three-dimensional volumes in position space or momentum space are denoted

$$\int d\vec{x} \quad \text{and} \quad \int d\vec{p}$$

while four-dimensional integrals are denoted

$$\int dx \quad \text{and} \quad \int dp$$

unless possibility of confusion requires

$$\int d^4x \quad \text{or} \quad \int d^4p$$

Lorentz vectors transform according to

$$x^{\mu} \xrightarrow[(a,\Lambda)]{} x'^{\mu} = \Lambda^{\mu}{}_{\lambda}\, x^{\lambda} + a^{\mu}$$

where (a,Λ) is an element of the Poincaré group with a^{μ} representing a translation in space-time and $\Lambda^{\mu}{}_{\lambda}$ representing a homogeneous Lorentz transformation.

The electromagnetic vector potential is

$$A^{\mu} = (\phi, \vec{A})$$

from which the field strengths

$$F^{\mu\lambda} = \partial^{\mu}A^{\lambda} - \partial^{\lambda}A^{\mu}$$

are obtained; and the electric and magnetic fields are given explicitly by

$$E^{i} = F^{i0} \qquad\qquad B^{i} = -\frac{1}{2}\, e_{ijk}\, F^{jk}$$

The completely antisymmetric tensors e_{ijk} and $e_{\alpha\beta\mu\lambda}$, $(i,j,k) = 1,2,3$ and $(\alpha,\beta,\mu,\lambda) = 0,1,2,3$, are determined by

$$e_{123} = 1 \quad \text{and} \quad e_{0123} = 1$$

The commutator of the operators a and b is denoted

$$[a,b] = ab - ba$$

while

$$\{a,b\} = ab + ba$$

is the anticommutator.

The Pauli matrices are

$$\sigma^{\mu} = (\sigma^{\circ}, \vec{\sigma}) = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]$$

and

$$\gamma^\mu = (\gamma^\circ, \vec{\gamma}) = \left[\begin{pmatrix} \sigma^\circ & 0 \\ 0 & -\sigma^\circ \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\right]$$

is the representation of the γ matrices used, satisfying the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, 1$$

and the relation

$$\gamma^\circ\, \gamma^{\mu\dagger}\, \gamma^\circ = \gamma^\mu$$

where the notation defined below has already been used.

The hermitian conjugate of an operator 0 is denoted

$$0^\dagger$$

while

$$\alpha^*$$

means the complex conjugate of a number or matrix α. The notation D^T is used for the transpose of a matrix D, and the hermitian adjoint matrix is defined

$$D^\dagger = D^{*T}$$

APPENDIX B

SL(2,c) AND ITS FINITE DIMENSIONAL, IRREDUCIBLE REPRESENTATIONS

SL(2,c) is the group of 2x2 unimodular matrices, and its homomorphism onto $L_+^\uparrow$ is given by (II-4). In the derivation of (II-4) and (II-5) use has been made of the following relations:

$$\mathrm{tr}\ \sigma^\mu \bar{\sigma}^\lambda = 2\ g^{\mu\lambda} \tag{B-1}$$

$$\sum_\mu (\sigma^\mu)_{ij} (\sigma^\mu)_{\alpha\beta} = 2\delta_{i\beta}\delta_{j\alpha} \tag{B-2}$$

Another useful relation of the above type is

$$\sum_\mu (\bar{\sigma}^\mu)_{ij} (\sigma^\mu)_{\alpha\beta} = 2\ C^{1/2}_{i\alpha}\ C^{1/2}_{j\beta} \tag{B-3}$$

the matrix $C^{1/2}$ being defined by (II-9).

Define the general element A of SL(2,c) as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

To find the representations of SL(2,c) we consider a two-dimensional vector

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The set S_{2j} of all monomials of degree 2j (2j a non-negative integer) in v_1 and v_2 transform among themselves under an arbitrary SL(2,c) transformation defined by

$$V \xrightarrow[A]{} V' = A\, V$$

The independent functions

$$e^j_m(V) = \frac{v_1^{\,j+m}\, v_2^{\,j-m}}{\sqrt{(j+m)!(j-m)!}} \qquad \begin{matrix} j=0,\ 1/2,\ 1\ \ldots \\ -j\le m\le j \end{matrix}$$

form a basis for the set S_{2j}, and matrices $D^j(A)$ can be defined by

$$e^j_m(AV) = D^j(A)_{mm'}\, e^j_{m'}(V) \tag{B-4}$$

The matrix elements $D^j(A)_{mm'}$ in (B-4), if solved for, are found to be

$$D^j(A)_{mm'} = \sum_k \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(m'-m+k)!(j-m'-k)!}\, a^{j+m-k}\, b^k\, c^{m'-m+k}\, d^{j-m'-k} \tag{B-5}$$

where the sum is over all integers k which do not produce factorials of negative numbers. These matrices are (2j+1)-dimensional and form irreducible representations of SL(2,c). For the general case the hermitian adjoint $D^j(A)^\dagger$ is not equivalent to $D^j(A)^{-1}$ since A is not generally unitary. However, for the unitary subgroup SU(2) the D-matrices are unitary and are the usual rotation D-matrices.* If R

*For the rotation D-matrices and Clebsch-Gordan coefficients we follow the phase conventions of Rose[25].

is in SU(2) we can show, by use of (II-6), that the complex conjugate representation $D^j(R)^*$ is equivalent to $D^j(R)$ by means of the relation

$$D^j(R) = C^j \, D^j(R)^* \, C^{j^{-1}} = D^j(R)^{-1\dagger} \tag{B-6}$$

where the matrix

$$C^j = D^j(-i\sigma^2) = D^j(R_2(\pi)) \tag{B-7}$$

has been introduced. The matrix elements of C^j are given by

$$C^j_{mm'} = (-)^{j+m}\delta_{m,-m'} \tag{B-8}$$

and C^j satisfies the relations

$$C^{j^T} = C^{j^{-1}} = (-)^{2j} \, C$$

All inequivalent, irreducible, finite-dimensional representations of SL(2,c) are obtained from direct products of the form (with S in SL (2,c))

$$D^{AB}(S)_{ab,a'b'} = D^A(S)_{aa'} \, D^B(S)^{-1\dagger}_{bb'} \tag{B-9}$$

where A and B separately assume the values 0,1/2,1,3/2,..., and these matrices multiply in the following manner:

$$D^{AB}(S_1)_{ab,\alpha\beta} \, D^{AB}(S_2)_{\alpha\beta,a'b'} = D^{AB}(S_1S_2)_{ab,a'b'}$$

These representations, except for the uninteresting A = B = 0 case, are nonunitary. For the SU(2) subgroup the D^{AB} matrices are unitary but are also reducible.

Direct products of the form $D^{j_1}(S)\, D^{j_2}(S)$ are reducible by use of the Clebsch-Gordon coefficients:

$$D^{j_1}(S)_{a_1a_1'}\; D^{j_2}(S)_{a_2a_2'} = \sum_{j_3,a_3,a_3'} \langle j_1a_1j_2a_2|j_3a_3\rangle\; D^{j_3}(S)_{a_3,a_3'}$$

$$\langle j_3a_3'|j_1a_1'j_2a_2'\rangle \qquad \text{(B-10)}$$

In (B-10) the j_3 sum is over all j_3 values satisfying the triangle inequality with j_1 and j_2, and the $\langle j_1a_1j_2a_2|j_3a_3\rangle$ are the Clebsch-Gordan coefficients satisfying the orthogonality and completeness relations

$$\sum_{m_1m_2} \langle jm|j_1m_1j_2m_2\rangle\; \langle j_1m_1j_2m_2|j'm'\rangle = \delta_{jj'}\delta_{mm'} \qquad \text{(B-11)}$$

$$\sum_{jm} \langle j_1m_1j_2m_2|jm\rangle\; \langle jm|j_1m_1'j_2m_2'\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'} \qquad \text{(B-12)}$$

From (B-10) and (B-11) there follows the important relation

$$D^A(S)_{aa'}\; D^B(S)_{bb'}\; \langle Aa'Bb'|jm\rangle = \langle AaBb|jm'\rangle\; D^j(S)_{m'm} \qquad \text{(B-13)}$$

As is evident from (B-5), for j = 1/2 the D-matrices are the elements of SL(2,c):

$$D^{1/2}(A) = A \qquad \text{(B-14)}$$

By use of (B-14), (II-5) and (II-8) can be written in the form

$$D^{1/2}(\Lambda)^{-1^T}(C^{1/2}\sigma^\mu)\ D^{1/2}(\Lambda)^\dagger = \Lambda_\lambda{}^\mu\ (C^{1/2}\sigma^\lambda) \qquad \text{(B-15)}$$

where the argument of the matrices is understood to be $A(\Lambda)$, i.e., the SL(2,c) matrix which induces the transformation Λ. Also, the relation (II-6)

$$(-i\sigma^2)\ A(-i\sigma^2)^{-1} = A^{-1^T}$$

can be written as

$$D^{1/2}((-i\sigma^2)\ A(-i\sigma^2)^{-1}) = D^{1/2}(A^{-1^T}) = D^{1/2}(A)^{-1^T}$$

which is a special case of the relation

$$C^j\ D^j(A)\ C^{j^{-1}} = D^j(A)^{-1^T} \qquad \text{(B-16)}$$

Some special D-matrices of interest are now given. If R is a member of SU(2) and induces the rotation $R_{\hat{n}}(\theta)$, a rotation of angle θ about the $\hat{n}$ axis, then the corresponding unitary D-matrix is

$$D^j(R_{\hat{n}}(\theta))_{mm'} = \left(e^{-i\theta\hat{n}\cdot\vec{J}^{(j)}}\right)_{mm'} \qquad \text{(B-17)}$$

where the $\vec{J}^{(j)}$ are the (2j+1)-dimensional angular momentum matrices. (As is the case frequently in this work, we have used the Lorentz transformation induced by the SL(2,c) matrix as the argument of the D-matrix in place of the SL(2,c) matrix itself.) Another special case occurs for those hermitian elements B of SL(2,c) which induce a boost $B(\vec{p})$ whose effect is to transform the rest momentum $P_R = (m,\vec{0})$ into the momentum $p^\lambda = (\omega,\vec{p})$ in a special way:

$$p^\mu = B(\vec{p})^\mu{}_\beta \, P_R^\beta$$

These special boosts are defined as

$$B(\vec{p}) = R(\hat{p}) \; B_3(|\vec{p}|) \; R^{-1}(\hat{p}) \tag{B-18}$$

where $B_3(|\vec{p}|)$ is a boost along the z-axis taking P_R to $(\omega,0,0,|\vec{p}|)$, and the rotation $R(\hat{p})$ takes $(\omega,0,0,|\vec{p}|)$ to $(\omega,\vec{p})$ in the manner defined by

$$R(\hat{p}) = R_3(\phi) \; R_2(\theta) \; R_3{}^{-1}(\phi) \tag{B-19}$$

where $R_2(\theta)$ is a rotation of angle θ about the y-axis, etc., and (θ,ϕ) are the angles defining $\hat{p}$. For these special boosts the D-matrices are

$$D^j(B(\vec{p}))_{mm'} = \left[e^{-\psi p\cdot\vec{J}^{(j)}}\right]_{mm'} \tag{B-20}$$

where $\cosh\psi = \omega/m$, $\sinh\psi = |\vec{p}|/m$, and these matrices are hermitian.

APPENDIX C

GRAPHICAL TECHNIQUES FOR CLEBSCH-GORDAN COEFFICIENTS

We begin by listing the completeness and orthogonality relations of the Clebsch-Gordan coefficients, along with some of their properties:

$$\sum_{jm} \langle j_1m_1j_2m_2|jm\rangle \langle jm|j_1m_1'j_2m_2'\rangle = \delta_{m_1m_1'}\,\delta_{m_2m_2'} \qquad \text{(C-1)}$$

$$\sum_{m_1m_2} \langle j'm'|j_1m_1j_2m_2\rangle \langle j_1m_1j_2m_2|jm\rangle = \delta_{jj'}\,\delta_{mm'} \qquad \text{(C-2)}$$

$$\langle jm|j_1m_1j_2m_2\rangle = (-)^{j_1+j_2-j} \langle jm|j_2m_2j_1m_1\rangle \qquad \text{(C-3)}$$

$$\frac{\langle jm|j_1m_1j_2m_2\rangle}{\sqrt{2j+1}} = (-)^{j_1-j+m_2} \frac{\langle j_1m_1|jmj_2-m_2\rangle}{\sqrt{2j_1+1}} \qquad \text{(C-4)}$$

$$\sum_{\lambda} \langle A_1a_1B_1b_1|j\lambda\rangle \langle j\lambda|A_2a_2B_2b_2\rangle$$

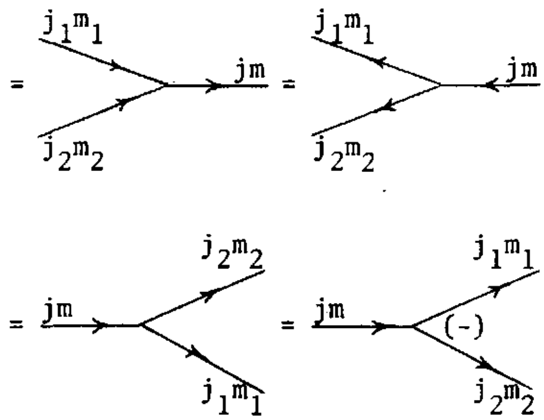
$$= (2j+1) \sum_{n,\nu} (-)^{A_1+B_1+A_2+B_2} \begin{Bmatrix} A_1 & B_1 & j \\ A_2 & B_2 & n \end{Bmatrix} \langle n\nu B_2b_2|A_1a_1\rangle \langle A_2a_2|n\nu B_1b_1\rangle \qquad \text{(C-5)}$$

We introduce the notation $[j] = 2j + 1$ and the function

$$\Delta(j_1j_2j_3) = \begin{cases} 1 & \text{if } |j_1-j_2| \le j_3 \le j_1+j_2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{(C-6)}$$

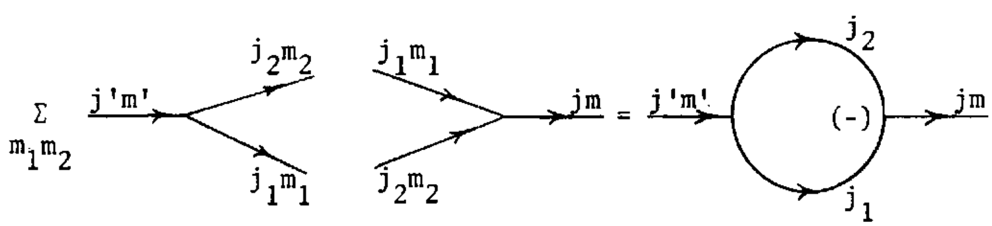
(This definition of Δ is not universal.) Clebsch-Gordan coefficients can be represented graphically in the following way:

$$\frac{\langle j_1m_1j_2m_2|jm\rangle}{[j]^{1/2}} =$$



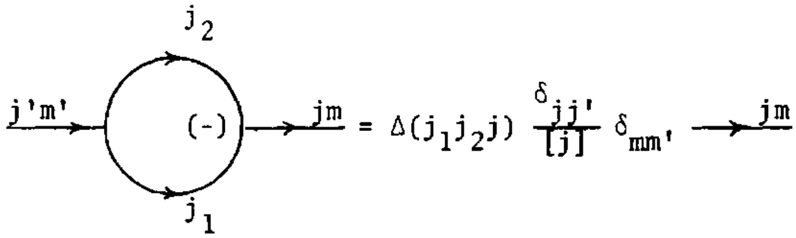
A graph with the indicator (-) at the vertex is read clockwise, and all graphs with no indicator are read counterclockwise. Each vertex must always have two ingoing lines and one outgoing line or one ingoing line and two outgoing lines. The two like-lines are coupled to give the third in the C-G coefficient. All arrows may be simultaneously reversed with no effect on the coefficient. (However, care must be taken when the graph is connected to something.)

Relation (C-2) may be expressed graphically as



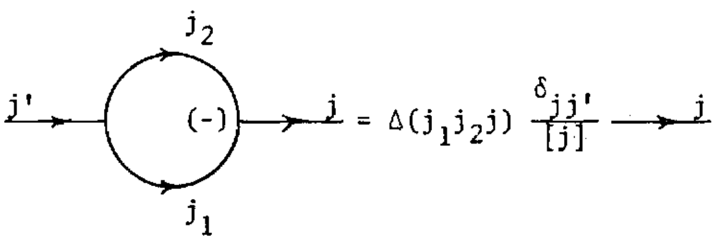
$$= \delta_{jj'}\,\delta_{mm'}\,\frac{\Delta(j_1j_2j)}{[j]}$$

or



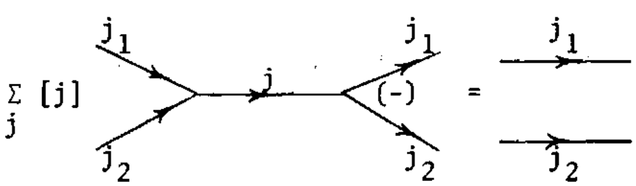
(C-7)

where the convention used is that an internal line represents a sum on the associated magnetic quantum number. Except where there is a possibility of confusion, we omit the magnetic quantum number labels, and (C-7) appears as



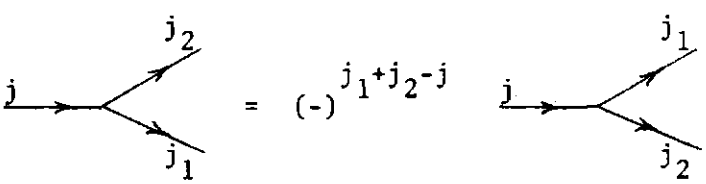
(C-8)

Relation (C-1) becomes

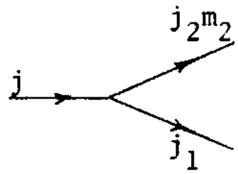


(C-9)

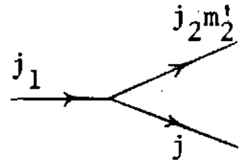
and (C-3) is



By use of the matrix C^j defined in (B-8), we can write (C-4) as

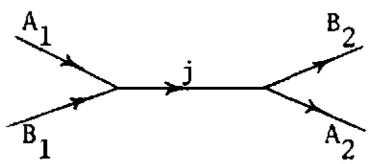


$= (-)^{j_1+j_2-j}$

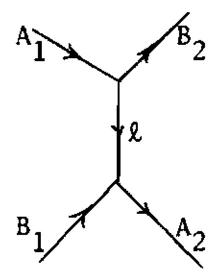


$C^{j_2}_{m_2' m_2}$ (C-10)

and relation (C-5) is given by

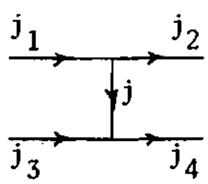


$= \sum_{\ell} [\ell] (-)^{A_1+B_1+A_2+B_2} \begin{Bmatrix} A_1 & B_1 & j \\ A_2 & B_2 & \ell \end{Bmatrix}$

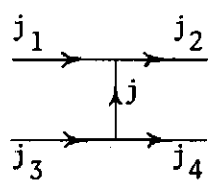


(C-11)

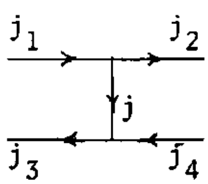
We can use (C-10) to find the rules for reversing internal lines:



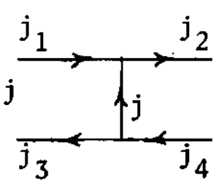
$=$



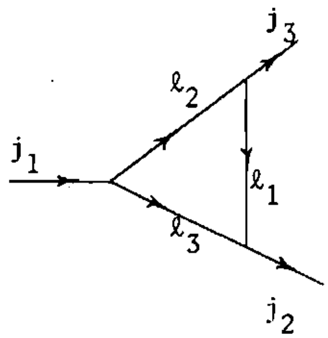
(C-12)



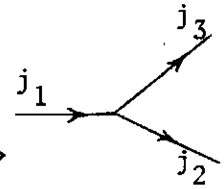
$= (-)^{2j}$



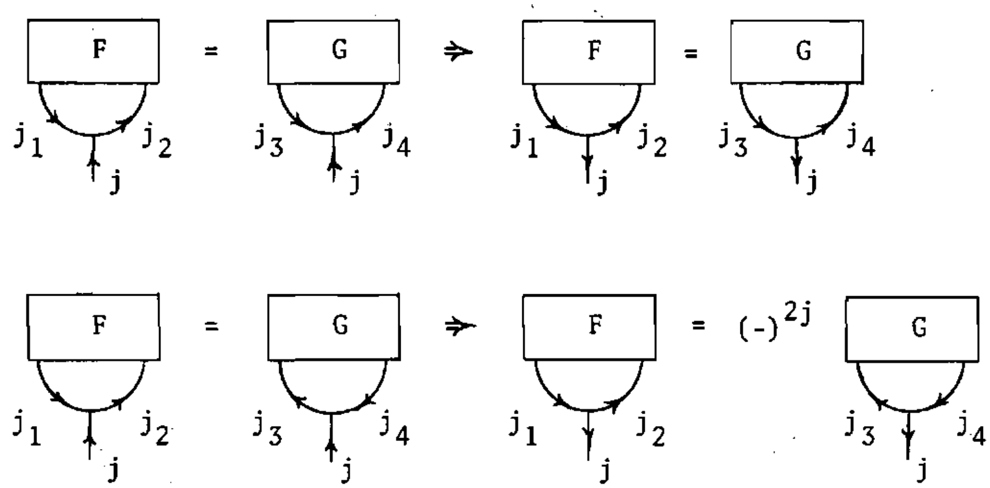
By use of (C-11) we can show a triangle collapses to a point in the following way:



$$= (-)^{j_1+j_2+j_3} \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix}$$

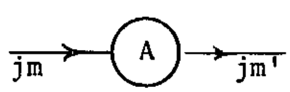


For generating additional relations, the following rules for reversing external lines in equations (obtained with the use of (C-10)) are useful:



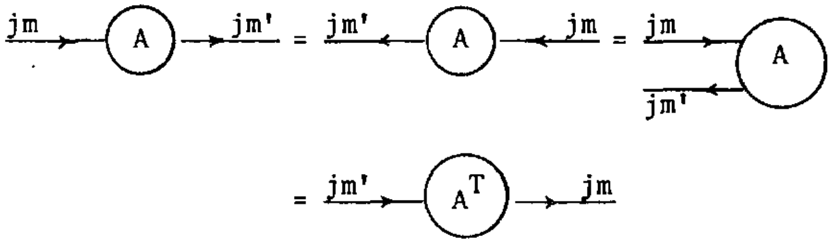
(C-13)

We need a graphical representation for matrices. This is easily obtained with the definition

$$D^j(A)_{mm'} =$$ 

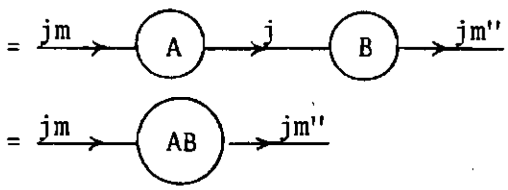
(C-14)

where the ingoing line carries the initial matrix subscript and the outgoing line carries the final matrix subscript. This definition results in such unusual looking relations as

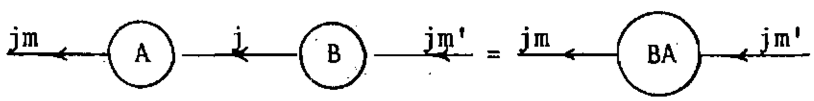


Matrix multiplication follows simply in the form

$D^j(A)_{mm'}\, D^j(B)_{m'm''}$ =

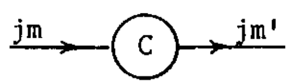


and care must be taken with such relations as

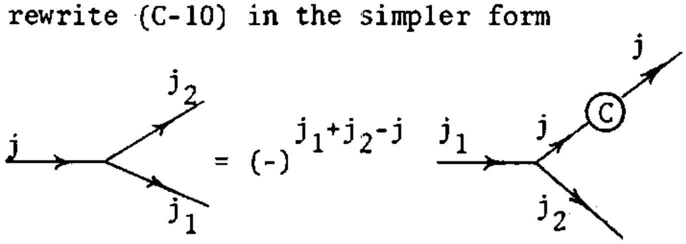


The matrix $C^j{}_{mm'} = D^j(-i\sigma^2)_{mm'} = D^j(C^{1/2})_{mm'}$ in this notation is

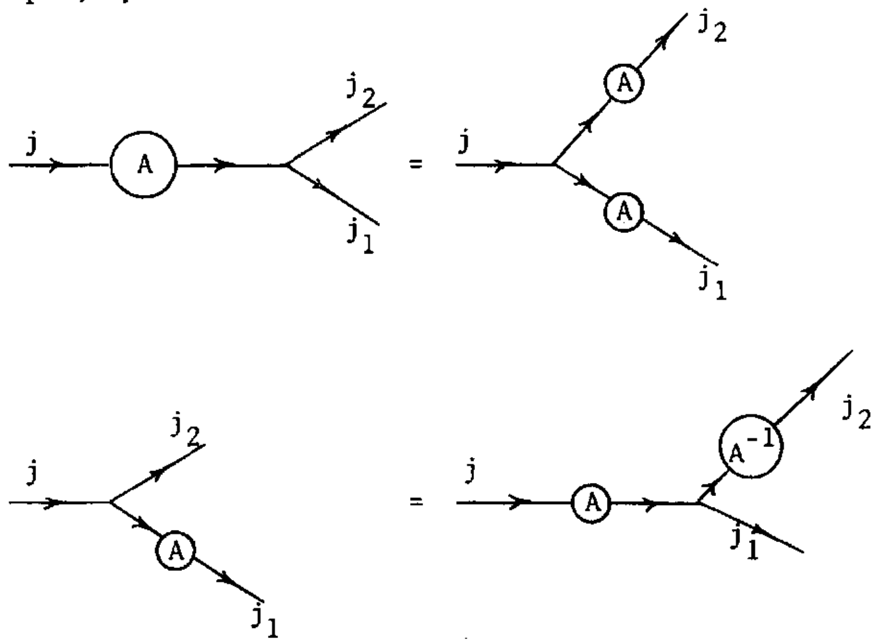
$C^j{}_{mm'}$ =



where the 1/2 superscript is left off of the SL(2,c) element $C^{1/2}$.
We may rewrite (C-10) in the simpler form



Relation (B-13), and similar ones obtained from it, may be represented, for example, by



(C-15)

Thus, the rules for moving matrices through vertices are:

(1) If the matrix follows a "continuous arrow flow" it is unaffected.

(2) If the matrix crosses an "arrow discontinuity" it is inverted.

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